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# Quantitative Techniques for the Assessment of Lake Quality

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Increases in wealth, mobility and scientific understanding have corresponded with an increase in our ability to influence the quality of the environment and in changes in our perception of environmental quality. The result has been a greater awareness of linkages between human activities and environmental degradation and an intensification of efforts devoted to enviromental management. This has been, and is being, demonstrated in the field of lake management.

Fundamental to the development of policies for lake management is the determination of the likely causal relationships among the lake and lake basin characteristics. This manual focuses on the role of phosphorus in lake eutrophication. If qualitatively presents the major causal linkages among the phosphorus-related components. Phosphorus was selected for this analysis because it is generally considered the most manageable of the major nutrients.

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# Quantitative Techniques For the Assessment of Lake Quality

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# QUANTITATIVE TECHNIQUES
# FOR THE ASSESSMENT
# OF LAKE QUALITY

by

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U.S. Environmental Protection Agency
Office of Water Planning and Standards

January 1979

# PREFACE

This manual provided a unique opportunity for me to discuss, in one volume, several related topics of research, including lake modeling, data analysis, and treatment of uncertainty. I am grateful to the Department of Resource Development, the Agricultural Experiment Station of Michigan State University, and to the Inland Lakes Management Unit of the Michigan Department of Natural Resources, for support and encouragement during this past year. I would also like to acknowledge the assistance of the following people from the Department of Resource Development, in the preparation of this document: Patty McKenna who typed draft versions of this report, Linda Boyer who typed the final copy, Steven Willis who edited the final version, and Paul Schneider who prepared much of the graphical material. Finally, I would like to thank Albert Massey, Richard Mikula, and Howard Wandell of the Inland Lakes Management Unit for helpful suggestions and criticisms of the draft report.

Kenneth H. Reckhow

East Lansing, Michigan
November, 1978

## TABLE OF CONTENTS

Page

PREFACE . . . . . . . . . . . . . . . . . . . . . . . . . . . ii

CHAPTER

1. INTRODUCTION . . . . . . . . . . . . . . . . . . . . . . . 1

2. DATA ANALYSIS AND SAMPLING DESIGN . . . . . . . . . . . 7

    2.1 Introduction . . . . . . . . . . . . . . . . . . . . 7

    2.2 Data Analysis . . . . . . . . . . . . . . . . . . . . 8

        2.2.1 Mean and Median . . . . . . . . . . . . . . . 8

        2.2.2 Standard Deviation and Variance . . . . . . 9

        2.2.3 Standard Error of the Mean . . . . . . . . . 12

        2.2.4 Correlation . . . . . . . . . . . . . . . . . 12

        2.2.5 Regression . . . . . . . . . . . . . . . . . 18

        2.2.6 Student's t . . . . . . . . . . . . . . . . . 21

    2.3 Sampling Design . . . . . . . . . . . . . . . . . . . 24

        2.3.1 Introduction . . . . . . . . . . . . . . . . 24

        2.3.2 Sampling Design Relationships . . . . . . . 26

        2.3.3 Estimation of Factors Important in Sampling Design . . . . . . . . . . . . . . . . . . . 33

        2.3.4 Some Specific Issues . . . . . . . . . . . . 34

        2.3.5 Annual Phosphorus Budget . . . . . . . . . . 35

        2.3.6 Summary and Conclusions . . . . . . . . . . 41

3. PHOSPHORUS MODELS . . . . . . . . . . . . . . . . . . . . 42

    3.1 Introduction . . . . . . . . . . . . . . . . . . . . 42

    3.2 Historical and Theoretical Development: A Brief Review . . . . . . . . . . . . . . . . . . . . . . . 42

CHAPTER Page

3.3 Empirical Models . . . . . . . . . . . . . . . . . . 47

3.4 Error Analysis . . . . . . . . . . . . . . . . . . . 56

3.5 A Case Study Comparison of Four Models . . . . . . 63

3.6 Special Lake Types . . . . . . . . . . . . . . . . . 73

3.7 Conclusions . . . . . . . . . . . . . . . . . . . . 78

Appendix . . . . . . . . . . . . . . . . . . . . . . . 79

Uncertainty Analysis . . . . . . . . . . . . . . . . 85

4. LAKE CLASSIFICATION . . . . . . . . . . . . . . . . . 89

4.1 Introduction . . . . . . . . . . . . . . . . . . . . 89

4.2 Trophic State Criteria . . . . . . . . . . . . . . . 90

4.3 Phosphorus Loading Criteria . . . . . . . . . . . . 96

4.3.1 Early Development . . . . . . . . . . . . . . 96

4.3.2 Recent Work by Vollenweider . . . . . . . . . 103

4.3.3 A Lake Classification Scheme Proposed by Uttormark and Wall . . . . . . . . . . . . 107

4.3.4 Uncertainty Analysis Applied to Vollenweider's Loading Criterion - A Graphical Approach . . . . . . . . . . . 110

4.3.5 Lake Quality Discriminant Analysis . . . . 120

4.4 Summary . . . . . . . . . . . . . . . . . . . . . . 127

Appendix . . . . . . . . . . . . . . . . . . . . . . . 130

Derivation and Explanation of the Procedure Used to Apply Uncertainty Analysis to Vollenweider's Loading Criterion . . . . . . . . . . . . . . . . . 130

Lake Quality Discriminant Functions . . . . . . . . 135

5. FINAL COMMENTS . . . . . . . . . . . . . . . . . . . 137

REFERENCES . . . . . . . . . . . . . . . . . . . . . . . 140

STATISTICAL SYMBOLS . . . . . . . . . . . . . . . . . . 144

SYMBOLS FOR LAKE MODELS . . . . . . . . . . . . . . . . 145

# LIST OF TABLES

TABLE Page

2.1 Total Phosphorus Concentration (mg/l) . . . . . . . . . . 8

2.2 Dissolved oxygen Concentration (mg/l) . . . . . . . . . . 11

2.3 Phosphorus Concentration (mg/l) in Two Hypothetical Lakes . . . . . . . . . . . . . . . . . . . . . . . . 22

2.4 Estimated Conditions in a Hypothetical Lake . . . . . . 30

2.5 Initial Uncertainty Estimates for Lake Winnipesaukee Phosphorus Loading . . . . . . . . . . . . . . . . . . . 40

3.1 Model Proposed by Reckhow (1977) for Oxic Lakes with $z/\tau < 50$ m/yr . . . . . . . . . . . . . . . . . . . . . . 52

3.2 Model Proposed by Walker (1977) . . . . . . . . . . . . 52

3.3 A Comparison of Empirical Models . . . . . . . . . . . 54

3.4 Data for Lake Charlevoix (EPA-NES, 1975) . . . . . . . 64

3.5 Comparison of the Predictions of the Four Models for Lake Charlevoix . . . . . . . . . . . . . . . . . . . . . 74

  3A.1 Models Developed by Reckhow (1977) . . . . . . . . 80

  3A.2 Statistics for 95 North Temperate Lakes (Reckhow, 1977) . . . . . . . . . . . . . . . . . . . 81

  3A.3 Statistics for the Three Model Classes . . . . . 82

  3A.4 Parameter Statistics for the Reckhow Models . . . 83

  3A.5 Parameter Statistics for the Walker Model . . . . 84

  3A.6 Prediction Error Contribution from Parameter Uncertainty for the Model Proposed by Reckhow for Oxic Lakes With $z/\tau < 50$ m/yr . . . . . . . . . . 84

4.1 Trophic State Vs. Chlorophy ll a (From EPA-NES 1974a) . 91

TABLE Page

4.2 EPA-NES Trophic State Delineation (From EPA-NES 1974a) . 91

4.3 Carlson's Trophic State Index . . . . . . . . . . . . . . 95

4.4 Permissible Loading Levels for Total Nitrogen and Total Phosphorus (Biochemically Active) ($g/m^2$-yr) (after Vollenweider, 1968). . . . . . . . . . . . . . . . . . . . 98

4.5 Point System for Lake Condition Index (From Uttormark and Wall, 1975). . . . . . . . . . . . . . . . . . . . . . 108

4.6 Suggested Management Strategies Associated with the Four Quadrants in Figure 4.5 (From Uttormark and Wall, 1975). . . . . . . . . . . . . . . . . . . . . . . . . . 111

# LIST OF FIGURES

FIGURE Page

1.1 Factors Affecting Lake Quality (From Dillon, 1974) . 2

2.1 A Normal Distribution and A Skewed Distribution . . . 10

2.2 The Relationship of Summer Average Chlorophyll to Spring Total Phosphorus (From Dillon and Rigler, 1974). . 14

2.3 The Relationship Between Median Total Phosphorus and Mean Chlorophyll *a* in Phosphorus Limited Lakes . . . 15

2.4 The Relationship Between Median Total Phosphorus and Mean Secchi Disc Depths in Phosphorus Limited Lakes . 16

2.5 Y versus X . . . . . . . . . . . . . . . . . . . . . 17

2.6 Sampling Design Uncertainty Versus Cost . . . . . . . 28

3.1 A Comparison of the Kirchner and Dillon Model for $R_p$ with Chapra's Constant Apparent Settling Velocity Approach . . . . . . . . . . . . . . . . . . . . . 50

3.2 Lake Ontario: Probability of Standard (P=.045 mg/l) Violation . . . . . . . . . . . . . . . . . . . . . 60

4.1 Vollenweider's Phosphorus Loading Criterion (1975), L versus $q_s$ $(z/\tau)$ . . . . . . . . . . . . . . . . . . 99

4.2 Dillon's Phosphorus Loading Criterion, With Lines of Constant Phosphorus Concentration (in μg/l) Distinguishing Trophic States (From Dillon, 1975) . . . . . 100

4.3 The Larsen-Mercier Phosphorus Loading Criterion (From Larsen and Mercier, 1975) . . . . . . . . . . . 102

4.4 Vollenweider's Phosphorus Loading Criterion (1976), L versus $q_s$, z, and $\tau$ . . . . . . . . . . . . . . . . . 106

4.5 Uttormark's and Wall's Lake Classification Scheme (From Uttormark and Wall, 1975) . . . . . . . . . . . 109

4.6 Uncertainty Estimates for the Loading Criteria on Vollenweider's 1975 Axes of L Versus $q_s$ (From Reckhow, 1978d) . . . . . . . . . . . . . . . . . . . . . . . 115

FIGURE Page

4.7 y Versus $q_s$ (From Reckhow, 1978d) . . . . . . . . . . . . 116

4.8 The Probability of Oligotrophic Classification as a Function of Phosphorus Loading, Model Uncertainty, and Phosphorus Loading Uncertainty (From Reckhow, 1978d). . 117

4.9 The Probability of Mesotrophic Classification as a Function of Phosphorus Loading, Model Uncertainty, and Phosphorus Loading Uncertainty (From Reckhow, 1978d). . 118

4.10 The Probability of Eutrophic Classification as a Function of Phosphorus Loading, Model Uncertainty, and Phosphorus Loading Uncertainty (From Reckhow, 1978d). . 119

4.11 Walker's Loading Criterion Assessing the Probabilities of Trophic Classification (From Walker, 1977) . . . . . 123

4.12 The Oxic-Anoxic Discriminant Function (From Reckhow, 1978a) . . . . . . . . . . . . . . . . . . . . . . . . 125

4.13 Lake Sammamish and Lake Washington: Probability of Oxic Conditions ($P_{oxic}$) vs. Annual Areal Phosphorus Loading (L) (From Reckhow, 1978a) . . . . . . . . . . . . 128

# CHAPTER 1

# INTRODUCTION

Increases in wealth, mobility and scientific understanding have corresponded with an increase in our ability to influence the quality of the environment and in changes in our perception of environmental quality. The result has been a greater awareness of linkages between human activities and environmental degradation and an intensification of efforts devoted to environmental management. This has been, and is being, demonstrated in the field of lake management. Motivated in part by the identified association between cultural activities and nutrient enrichment, work on state and federal programs (such as those funded under Sections 208 and 314 of P.L. 92-500) may provide insight on successful lake management strategies.

Fundamental to the development of policies for lake management is the determination of the likely causal relationships among the lake and lake basin characteristics. This manual focuses on the role of phosphorus in lake eutrophication. Figure 1.1 (Dillon, 1974) qualitatively presents the major causal linkages among the phosphorus-related components. Phosphorus was selected for this analysis because it is generally considered the most manageable of the major nutrients. Further, as Sawyer (1971) notes:

> The phosphorus content of domestic and certain industrial wastewaters has been closely scrutinized for at least three

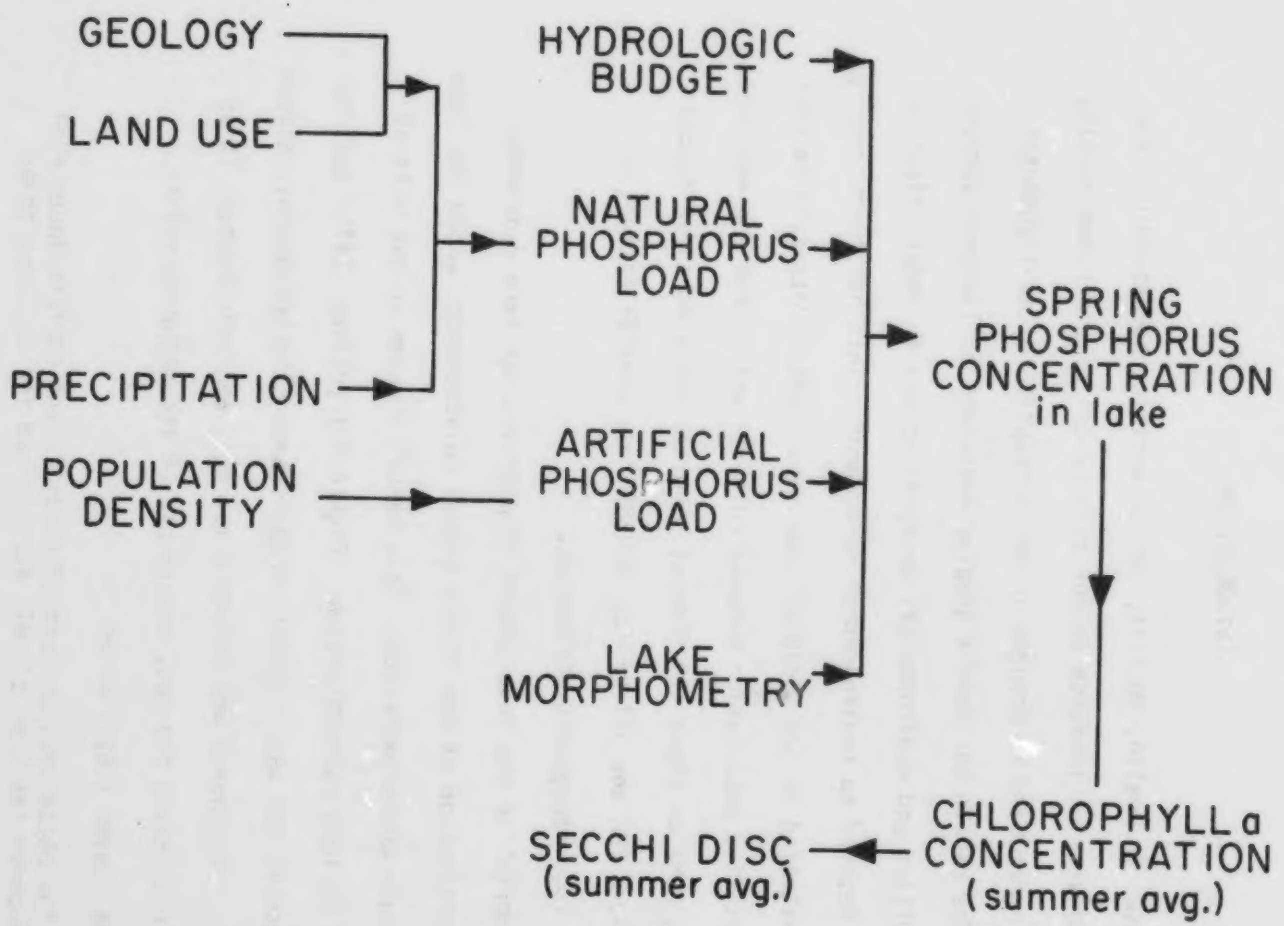
GEOLOGY
LAND USE
PRECIPITATION
POPULATION DENSITY
HYDROLOGIC BUDGET
NATURAL PHOSPHORUS LOAD
ARTIFICIAL PHOSPHORUS LOAD
LAKE MORPHOMETRY
SPRING PHOSPHORUS CONCENTRATION in lake
CHLOROPHYLL a CONCENTRATION (summer avg.)
SECCHI DISC (summer avg.)

Figure 1.1 Factors Affecting Lake Quality (From Dillon, 1974)

reasons. First, wastewater treatment methods to remove phosphorus from the effluent have been known for several years. Second, a large part of the phosphorus in domestic wastewaters and essentially all phosphorus in some industrial wastes is contributed by synthetic detergents. Third, phosphorus limitation in lakes and streams seems to be the only known means to control the nitrogen-fixing blue-green algae. Thus, phosphorus removal must be part of any plan to control cultural eutrophication.

In order to successfully design a wastewater treatment plant for phosphorus removal, one needs to quantify the inputs to the proposed plant and use mathematical equations, or models, from the sanitary engineering literature to specify the design. Clearly, little faith would be put in a design that resulted solely from intuitive judgment with no consideration of the quantitative information available. Similarly, the evaluation of strategies for lake management can and should reflect consideration of quantitative information on lake inputs (of phosphorus, for instance) together with mathematical equations, or models, that relate inputs to lake quality. Here, too, it would not be wise to rely solely on intuitive judgment. However, personal judgment of people experienced in the management of the system of interest (a lake, for example) can be quite important in the design of a data gathering program, in the interpretation of the data, and in the interpretation of a model's prediction. This is discussed in Chapter 2.

Additional justification for the use of all available sources of input information, particularly those that are quantitative in nature, comes from the realization that all decisions imply a weighing of costs, benefits, and other characteristics affected by the decision. Whatever the adopted strategy, the implication is that the resultant quality associated with that strategy is somehow optimal given the costs, benefits and constraints. Thus even though a decision maker may resist quantification or cost-benefit comparisons because it does not seem reasonable to him/her, any management strategy selected has this quantification or weighing of factors implicitly written in. It seems more reasonable to attempt to quantify all relevant factors, weigh or relate them (through a model, perhaps) and thus explicitly consider this quantitative information as one input to the decision-making process.

This is not to be taken as blanket endorsement of quantification and modeling, however. It is critical that an appropriate model be selected (assuming an appropriate model can be found) for the intended purpose, and that the individual applying the model has a clear understanding of the model's limitations. Lake models exist over a wide range of complexity and cost of application. As a rule, the highly complex theoretically-based models are intended to study interactions among ecosystem components and are not meant for general management application. Their values lie in increasing knowledge of the chemical, physical, and biological interactions. However, their cost is prohibitive and their accuracy questionable (for the short time frame changes they are designed to examine) for general use. On the other hand, simple empirical models, like those discussed in the following chapters, are designed to assess the quality impact of a single component. They require relatively little data and are easily employed using a "desk-top" calculator. However, since they are empirical they cannot be applied with confidence beyond the range of conditions in the lakes used to establish the model's parameters. For instance, if the data set used to develop the model contained lakes with a mean depth range of five to sixty meters, then the model results may not hold for lakes with mean depths outside this range (and as is shown in Chapter 3, the confidence limits widen considerably within this range as the extremes are approached). Another example of this empirical model limitation is noted in Chapter 3 for the Dillon-Rigler-Kirchner model. This model was developed from a phosphorus-poor data set from the Canadian Shield, and thus it must be applied with caution in nutrient-enriched lakes.

It should be the modeler's responsibility to clearly document the proper use and limitation on the use of his/her model. For an empirical model, this documentation should include a statement of all limitations (geographic, trophic, geomorphologic, etc.) associated with the data set used to develop the model. In addition, any biases noted within the range of application should also be indicated. Another important but generally neglected statement of information about the model concerns the type of issues and decisions appropriate for model application and the value of the information provided by the model toward this application. As noted above, a variety of lake models exist, and a particular model or type of model may be justified as most suitable for a given application. The modeler will enhance the value of the model as a management tool if a clear statement is provided discussing appropriate application.

A measure of the value of the information provided by the model may be obtained from an explicit consideration of uncertainty. Model error can be combined with input variable error for the assignment of confidence limits to the prediction using first-order analysis. This concept has not previously been considered in the application of models for lake management, so it is particularly important here. Prediction error provides a direct measure of model prediction information; as the confidence limits associated with a prediction become smaller, a decision-maker should assign more weight to the information provided by the model.

When the prediction uncertainty is calculated, it may easily be combined with a probabilistic water quality standard (e.g., a 90% chance that the phosphorus concentration is less than .025 mg/l) to assess the likelihood of a standard violation. Few standards are presently stated in a probabilistic form. However, since neither mathematical nor conceptual projections of future conditions can be made with certainty, a standard expressed as a probability of violation is consistent with our ability to characterize the system. An example illustrating how a probabilistic standard facilitates decision making is presented in Chapter 3.

The remainder of this manual outlines quantitative techniques that can be used to evaluate strategies for lake management. Chapter 2 presents a number of commonly employed statistical techniques for the design of sampling programs and for data analysis. Chapter 3 discusses the simple empirical models for phosphorus, and Chapter 4 outlines several schemes proposed for lake classification. Examples are presented to illustrate the use of the techniques outlined in those chapters. In conclusion, Chapter 5 briefly suggests additional research that may lead to refinements of the techniques described herein.

# CHAPTER 2

# DATA ANALYSIS AND SAMPLING DESIGN

## 2.1 Introduction

Data analysis and statistics are becoming more common in water quality studies. Often statistics (e.g., the mean) are used to summarize the information from a set of data, and thus it is important to interpret these statistics properly. Statistical tests may be used to evaluate or compare data sets, and thus we need to know how to apply the tests and use the results. These topics are addressed in this chapter.

One area of major importance in water quality studies in which statistical analysis has not been used to full advantage is the design of sampling programs. All too often, the selection of water quality sampling sites and times is based on convenience (e.g., sample once every two weeks at an easily-reached site), and no consideration is given to the value of the information gathered. As we shall see, sampling programs should reflect the variance (uncertainty) of the populations sampled and the cost of sampling. Convenience, if it does enter into the design, is a cost factor, but it must often be balanced against a reduction in the value of the data gathered. This topic is considered later in this chapter.

## 2.2 Data Analysis

### 2.2.1 Mean and Median

The mean, or average value, is defined as the sum of a set of data divided by the number of elements:

$$\text{mean} = \bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad (2.1)$$

where $\sum_{i=1}^{n}$ = the sum of the elements $x_i$ from $i=1$ to $i=n$.

The median is the middle value (or the average between the two middle values if the data set contains an even number of data points) in a set of data when the data are ranked from lowest to highest (or vice versa).

An example provides a good comparison of the information presented by each of these statistics representing the "central tendency" of a set of data. Table 2.1 presents the total phosphorus concentration taken in a lake at one site at five different depths (top to bottom).

TABLE 2.1

TOTAL PHOSPHORUS CONCENTRATION (mg/l)

| Data | Median | Mean |
|---|---|---|
| .020 | .022 | .040 |
| .022 | | |
| .021 | | |
| .042 | | |
| .095 | | |

The median, or middle value is .022 mg/l, while the mean is .040 mg/l. This points out an important distinction between these two statistics. The mean is influenced by extreme values (e.g., .095 mg/l) while the median is not. To view this in a general sense, look at Figure 2.1. The horizontal axis presents numerical values (such as the data in Table 2.1), and the vertical axis presents the frequency of occurrence of each numerical value in the data set. Two curves that represent the distribution of numerical values in hypothetical sampling programs are displayed. When the data are distributed "normally" (a bell-shaped curve), the median and the mean are equal. However, when the data are skewed (some extreme values exist), the mean will be influenced by the extreme values in the skewed distribution and will differ from the median.* It is quite important to realize that this distinction exists between the mean and the median. For this reason, it is informative to present additional information on a data set, reflecting the spread (variance) and the skewness. In addition, when examining a report in which only the mean or median is presented, be aware that your ability to determine the characteristics of the original data (or the population) is limited.

### 2.2.2 Standard Deviation and Variance

The standard deviation describes the spread in a set of data. Mathematically it is estimated by:

---

*A commonly observed skewed distribution is the log-normal, which looks similar to the skewed distribution in Figure 2.1. It is called log-normal because a logarithmic transformation of all data results in a normal distribution. The mean of the logarithmically-transformed data equals the geometric mean of the original data set.

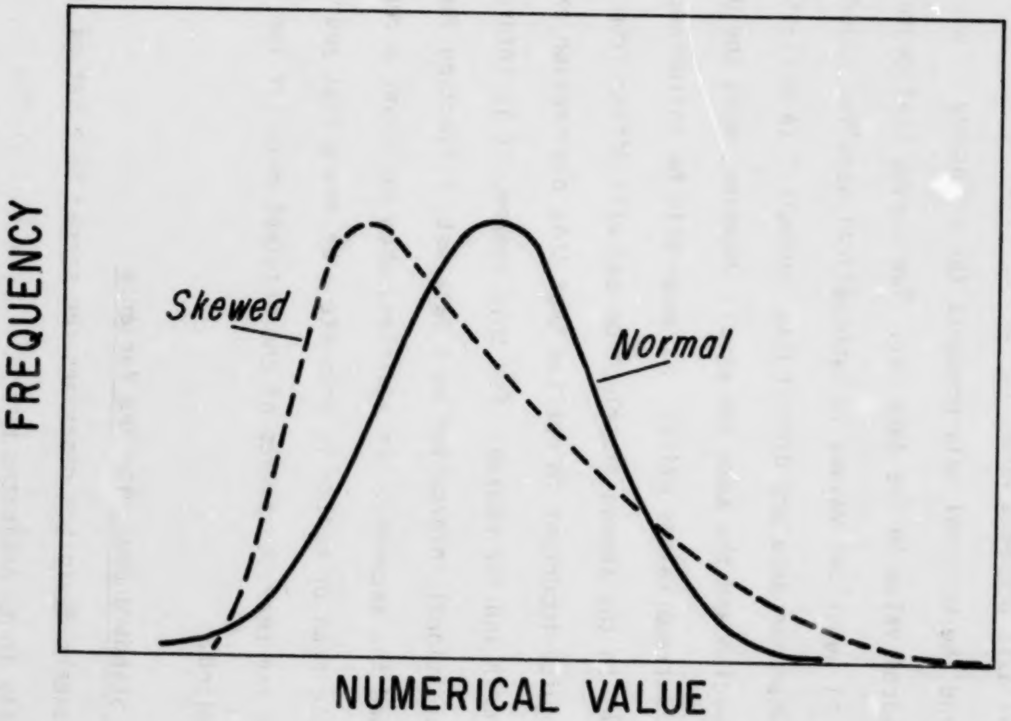

Figure 2.1 A Normal Distribution and A Skewed Distribution

$$\text{standard deviation} = s = \left[ \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2 \right]^{\frac{1}{2}} \qquad (2.2)$$

The variance is simply the standard deviation squared.

As an example, these statistics will be estimated for the data in Table 2.2, which presents dissolved oxygen data for the epilimnion of a lake. Note that the mean and the median are nearly identical as the set of data in Table 2.2 has no extreme values.

TABLE 2.2

DISSOLVED OXYGEN CONCENTRATION (mg/l)

| Data | Mean | Standard Deviation |
|---|---|---|
| 8.3 | 8.2 | 0.33 |
| 7.9 | | |
| 7.9 | | |
| 8.0 | Median | Variance |
| 8.7 | | |
| 8.1 | 8.1 | 0.11 |
| 8.6 | | |

A good way to describe this data set is: "approximately uniformly distributed with a mean of 8.2 mg/l ± 0.33 mg/l." Therefore, a good presentation of data should always include the standard deviation (or another measure of data "spread"), and graphical presentation should include error bars (± one standard deviation) on mean values. If a data set is normally distributed, about 2/3 of the data fall within $\pm s$ of the mean.

The standard deviation or the variance provides a good measure of the value of the information in a set of data. Specifically, the inverse of the variance has been equated to "information value" (M. G. Kendall and A. Stuart, 1967). Alternately, the variance is an estimate of uncertainty, a concept that is mentioned in Chapter 1 and used to assess the impact of lake management strategies in Chapter 3 and Chapter 4.

### 2.2.3 Standard Error of the Mean

The standard error of the mean expresses the uncertainty, or conversely the confidence, in the estimate of the mean value for a set of data. It is estimated by:

$$s_{\overline{x}} = \frac{s}{\sqrt{n}} \tag{2.3}$$

Note that the relationship equates the standard error of the mean to the standard deviation divided by the square root of the number of samples. This means that our uncertainty in the estimate of the mean is less than the uncertainty in the data set (represented by the standard deviation). Further, the larger the number of samples, the better (less error) we can estimate the mean.

### 2.2.4 Correlation

Correlation may be defined (statistically) as a measure of linear association. It is expressed in terms of a correlation coefficient, r, which may be calculated by:

$$r = \frac{\sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \overline{x})^2 \sum_{i=1}^{n} (y_i - \overline{y})^2}} = \frac{\text{covariance }(x,y)}{s(x)\ \ s(y)} \tag{2.4}$$

r ranges between +1 and -1, and the higher the absolute value of r, the stronger the linear association. Figures 2.2 - 2.4 present plots of bivariate data, all exhibiting relatively high values of the correlation coefficient. Note the difference in slope between the positive coefficient in Figure 2.3 and the negative coefficient in Figure 2.4, and the difference in the strength of the association between Figure 2.2 ($r=.95$) and Figure 2.3 ($r = .78$).

In the application of Equation 2.4 or the evaluation of a correlation coefficient and its associated bivariate plot, there are a few concepts that must be kept in mind.

1. The correlation coefficient evaluates a linear (straight line) relationship. Often, a transformation (e.g., logarithmic) will turn what appears to be a nonlinear relation into a linear one. However, the correlation coefficient will not reflect an intrinsically nonlinear relationship.

2. The correlation coefficient is likely to be biased towards a higher absolute value if it is applied to a set of data that is not normally distributed. For example, Equation 2.4 "sees" the set of data in Figure 2.5 as basically 4 points (one large point at the origin) with a fairly strong linear association. As a result, each of the three extreme points on the right has more weight in determining the correlation coefficient than does any one point in the bunch at the lower left. Thus, error in the point at the extreme right can significantly affect the correlation coefficient. A transformation of x and y, that results in a normal distribution

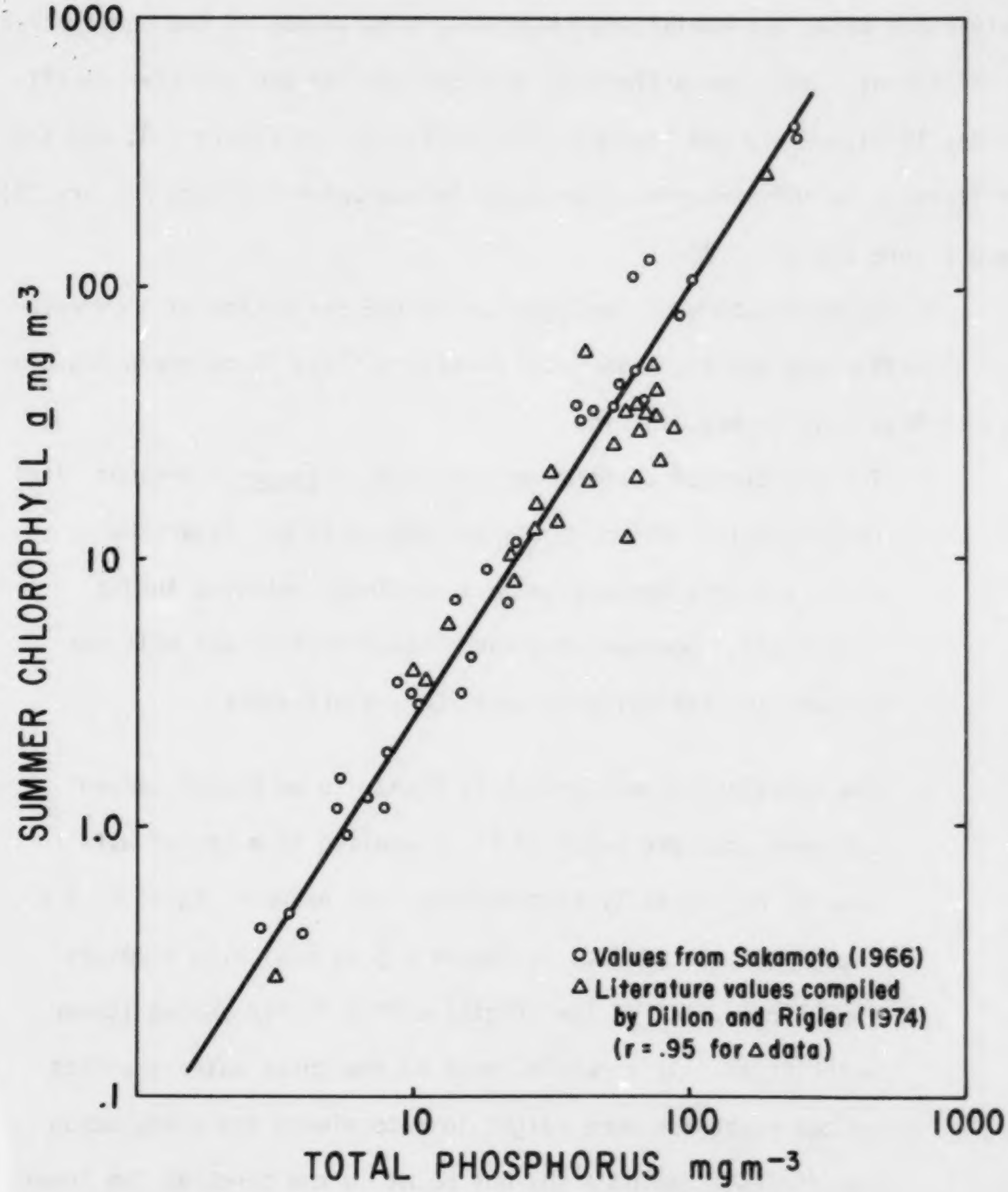

Figure 2.2 The Relationship of Summer Average Chlorophyll to Spring Total Phosphorus (Dillon and Rigler, 1974)

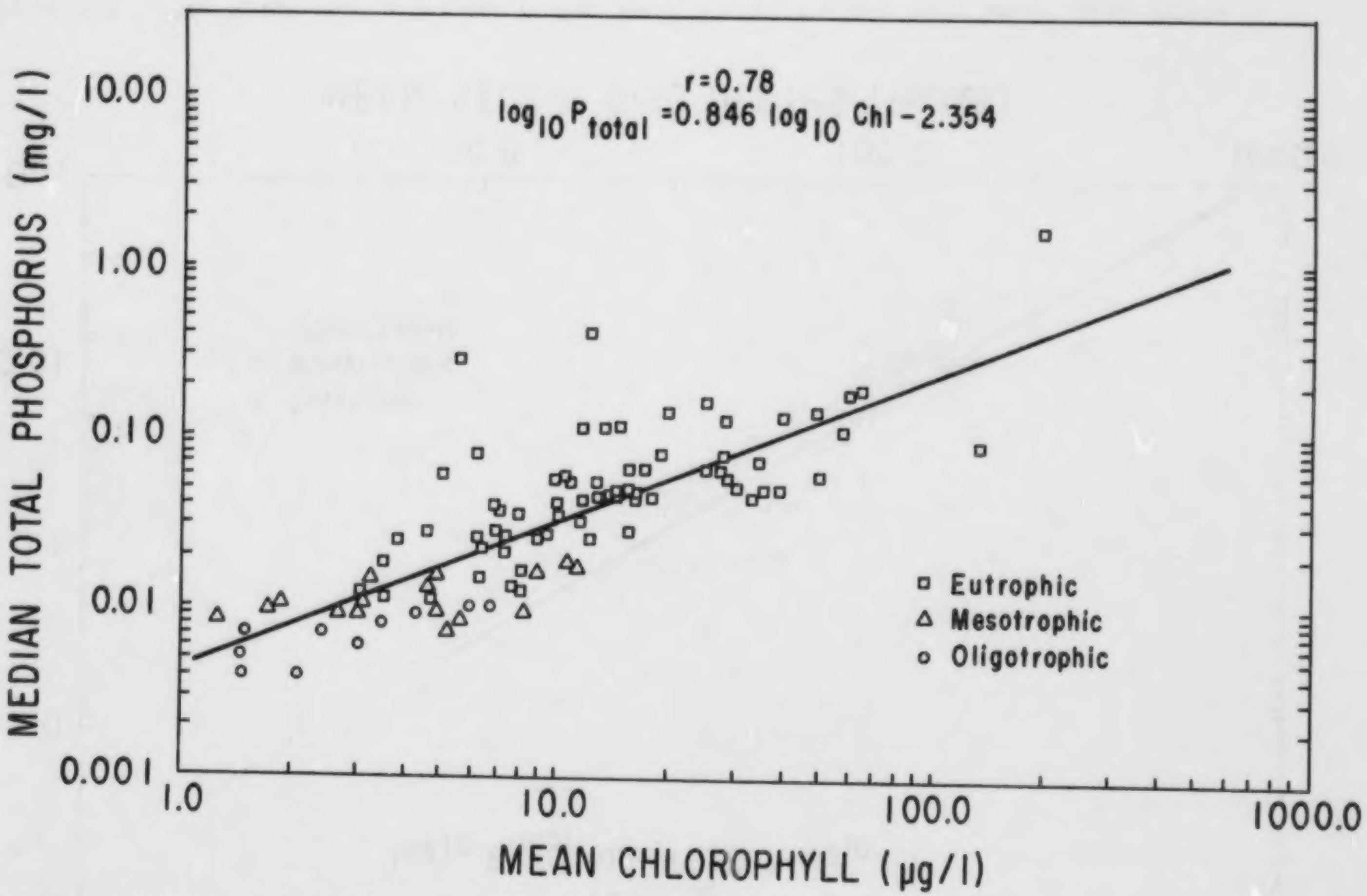

Figure 2.3 The Relationship Between Median Total Phosphorus and Mean Chlorophyll $\underline{a}$ in Phosphorus Limited Lakes

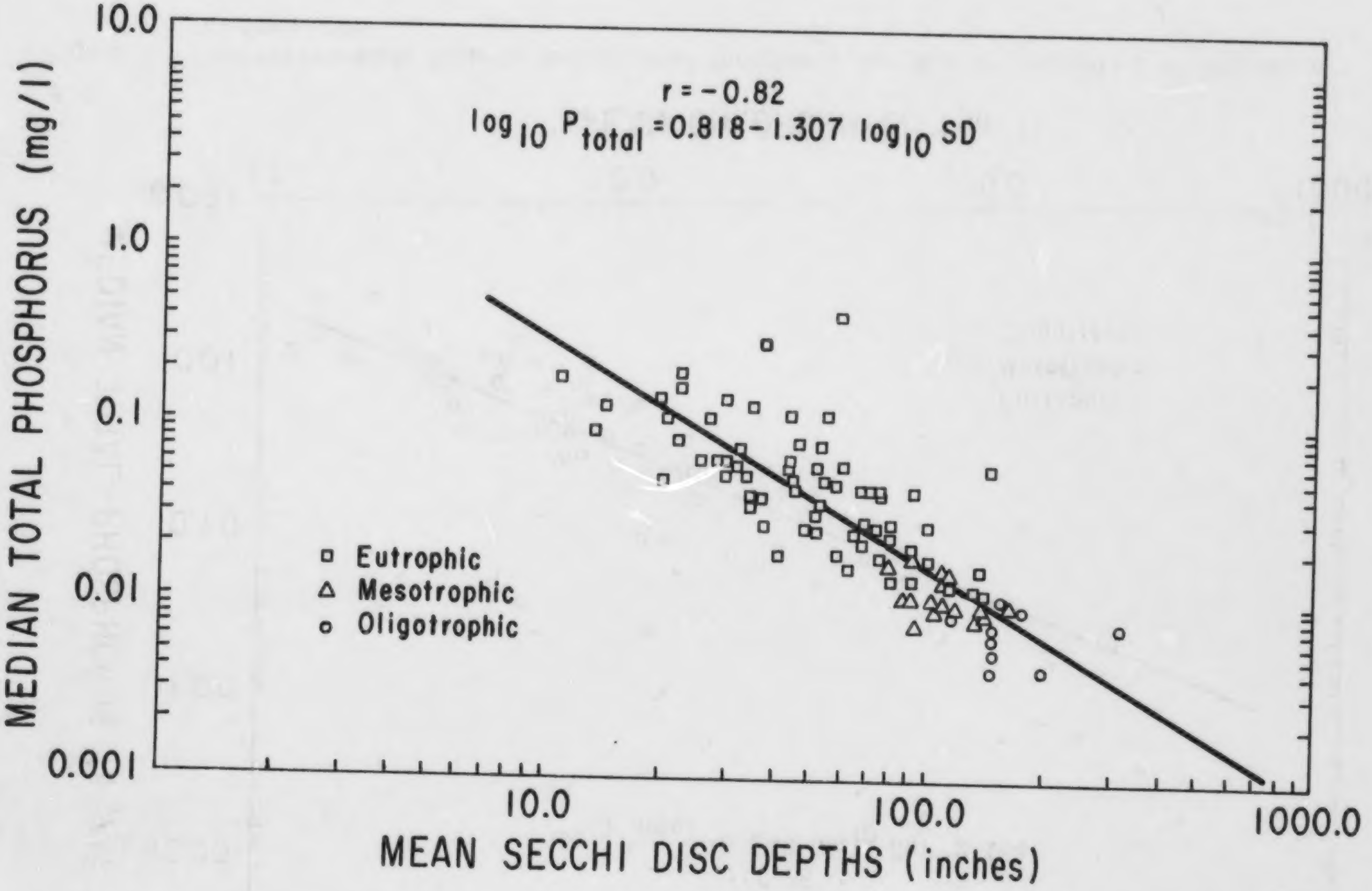

Figure 2.4 The Relationship Between Median Total Phosphorus and Mean Secchi Disc Depths in Phosphorus Limited Lakes

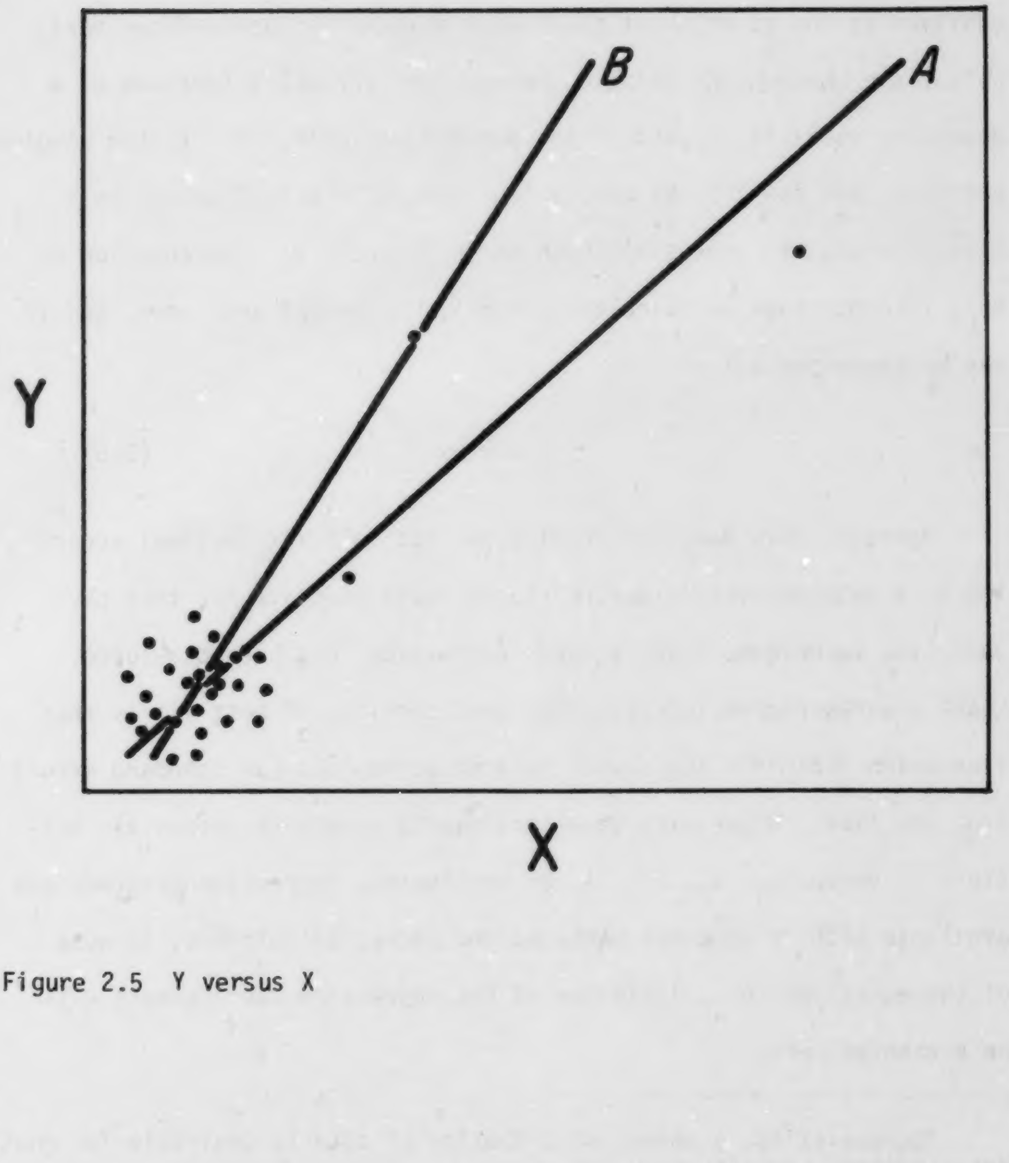

Figure 2.5 Y versus X

of data along the x-axis and along the y-axis, leads to the desired weighting of the data points in determining the correlation coefficient.*

### 2.2.5 Regression Analysis

There are many situations in which it is desirable to know, in addition to the strength of the linear association between two variables, the functional relation between the variables (defined as a dependent variable, y, and an independent variable, x). In the simplest approach, one can fit, by eye, a line through a set of points on a bivariate plot (y versus x), such as in Figure 2.5. The equation of this line may then be calculated from its intercept and slope, and it may be expressed as:

$$y = a + bx \qquad (2.5)$$

However, this does not provide an "optimal" fit (optimal according to a mathematical criterion) to the data points. For that purpose, the technique, least squares regression, has been developed. Least squares regression specifies that the line of best fit is that line which minimizes the sum of squared deviations (or standard error) from the line. Regression equations may be simple (y versus x), multiple (y versus $x_1$, $x_2$, . . .), or nonlinear. Regression programs are available both in computer packages and pocket calculators, so none of the equations for calculation of the regression coefficients will be presented here.

---

*Generalizing, a normal distribution of data is desirable for most data analysis techniques.

Two measures of "goodness of fit" of a regression equation are the coefficient of determination, $R^2$, and the standard error of the estimate, $s_m$. $R^2$ is a measure of the fraction of the variation in y "explained" by the regression equation. It is also the square of the correlation coefficient between the predicted and the observed values. $s_m$ may be estimated by:

$$s_m = \sqrt{\frac{\Sigma(Y_{pred.} - Y_{obs.})^2}{d.f.}} \qquad (2.6)$$

where:

$Y_{pred.}$ = predicted Y value

$Y_{obs.}$ = observed Y value

d.f. = degrees of freedom

Both $s_m$ and $R^2$ may provide biased estimates of the goodness of fit if the data are not normally distributed. For example, if $R^2$ and $s_m$ were estimated from the untransformed data in Figure 2.5, $R^2$ would be higher, and $s_m$ lower, than their "true" values. The regression between y and x in Figure 2.5 should occur under variable transformations (perhaps logarithmic) that yield approximately normally distributed variables (as in Figures 2.3 and 2.4, for example).

$s_m$ is required for the estimation of "confidence limits" on a regression line. Confidence limits on a simple regression line are estimated by:

$$\text{Confidence limits} = Y_{\text{pred.}} \pm ts_m \sqrt{\frac{1}{n} + \frac{(x_o - \overline{x})^2}{\Sigma(x_i - \overline{x})^2}} \qquad (2.7)$$

where:

$t$ = student's t (discussed in next section)

$\overline{x}$ = mean of value of x

$n$ = number of data points for x, y

$x_o$ = a particular value of x

Solution of Equation 2.7 for a range of $x_o$-values will provide confidence limits for the line at the desired significance level (specified through t). These confidence limits may be interpreted such that the true line will fall within $Y_{\text{pred.}}$ ± the confidence limits at a frequency specified by the significance level (e.g., 95% of the time, 70% of the time, etc.).

Often confused with the confidence limits on the line are the confidence limits on a prediction, or the "prediction interval." The prediction interval is estimated by:

$$\text{Prediction interval} = Y_{\text{pred.}} \pm ts_m \sqrt{1 + \frac{1}{n} + \frac{(x_o - \overline{x})^2}{\Sigma(x_i - \overline{x})^2}} \qquad (2.8)$$

The prediction interval is generally much larger than the confidence limits, as it reflects the variability in a single data point. Prediction intervals are used to estimate prediction uncertainty for the models discussed in Chapters 3 and 4.

Regression analysis is an extremely valuable tool in describing variable relationships and in developing predictive equations. However, a few final words of caution are in order:

1. Multicollinearity, or correlation among independent variables, can affect the regression coefficients. Thus, cause-effect conclusions are difficult when multicollinearity is significant.

2. Regression equations are empirical, and thus they should not be applied outside the range of conditions that exist in the model development data set.

3. Significant errors in the independent variables can bias the model parameters.

4. When developing or using a regression equation, it is a good rule to check for normally distributed variables (required if confidence limits or prediction intervals are to be used). If the variables are not normally distributed, transform the variables to achieve a normal distribution. Be wary of the interpretation of a regression equation developed from variables that are not normally distributed.

### 2.2.6 Student's t

Student's t, presented in Equations 2.7 and 2.8, is a useful statistic for significance tests with small samples. Most statistics books and some calculators provide values of t as a function of the number of degrees of freedom and the specified significance level.

A particularly common use of student's t is the t-test for the comparison of two groups of data. For example, to test the "null" hypothesis that there is no significant difference between the mean values of phosphorus concentration in two lakes, consider the hypothetical data in Table 2.3.

TABLE 2.3

PHOSPHORUS CONCENTRATION (mg/l) IN TWO HYPOTHETICAL LAKES

| Lake A | | Lake B |
|---|---|---|
| .010 | | .015 |
| .012 | | .017 |
| .015 | | .013 |
| .006 | | .011 |
| .007 | | .015 |
| .010 | | .011 |
| .010 | | .016 |
| .013 | | .014 |
| .017 | | .019 |
| .009 | | .009 |
| .011 | | .010 |
| .0109 | mean | .0136 |
| .120 | sum | .150 |

Student's t is calculated from:

$$t = \frac{\overline{x}_A - \overline{x}_B}{\sqrt{\frac{s^2(n_A + n_B)}{(n_A)(n_B)}}} \qquad (2.9)$$

where:

$\overline{x}_A, \overline{x}_B$ = the means for groups A and B

$n_A, n_B$ = the number of samples in groups A and B

$s^2$ = the pooled within - group variance
(calculated below, for the example)

For the problem posed, first calculate the sums of squares (ss) within each group:

$$ss_A = \Sigma x_A{}^2 - \frac{(\Sigma x_A)^2}{n_A} = (.010)^2 + (.012)^2 + \ldots + (.011)^2 - \frac{(.120)^2}{11}$$

$$ss_A = 1.049 \times 10^{-4}$$

$$ss_B = \Sigma x_B{}^2 - \frac{(\Sigma x_B)^2}{n_B} = (.015)^2 + (.017)^2 + \ldots + (.010)^2 - \frac{(.150)^2}{11}$$

$$ss_B = 9.855 \times 10^{-5}$$

The pooled within - group variance is:

$$s^2 = \frac{ss_A + ss_B}{(n_A - 1) + (n_B - 1)}$$

$$s^2 = \frac{1.049 \times 10^{-4} + 9.855 \times 10^{-5}}{10 + 10}$$

$$s^2 = 1.017 \times 10^{-5}$$

Thus:

$$t = \frac{.0109 - .0136}{\sqrt{1.017 \times 10^{-5}\left(\frac{11 + 11}{(11)\ (11)}\right)}}$$

$$t = - 1.985$$

This value of t has $(n_A-1) + (n_B-1)$, or 20, degrees of freedom. Consulting a t-table, we find that for 20 degrees of freedom, this value of t is significant at the 90% level, but not quite at the 95% level. Thus, depending upon the acceptable level of significance (i.e., the risk of being wrong in one direction), we can reject (at 90%) or accept (at 95%) the null hypothesis (that there is no significant difference in phosphorus concentration between the lakes).

In conclusion, some words of caution on the t-test are in order:

1. The samples should be random (or relatively random, see the next section on sampling design).

2. The population (e.g., all possible phosphorus concentration values in the lake of study) should be normally distributed. A transformation (e.g., logarithmic) may be necessary.

3. The t-test, as applied, is based on the assumption of equal variance for the groups (lakes) compared.

## 2.3 Sampling Design

### 2.3.1 Introduction

The gathering of water quality samples is a task that requires careful planning, or design. Sampling should always be undertaken with an objective in mind, whether the sampling provides "baseline" data, nutrient flux from a particular source, or a nutrient budget for a lake. In general terms, the objective may usually be stated as: minimize the uncertainty in the estimate of the characteristic(s) of interest subject to a budget (cost) constraint.*

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*Conversely, sampling may be undertaken to estimate the characteristic(s) of interest to a specified level of precision (uncertainty) at a minimum cost.

Data collection provides a limited amount of information (a sample) about a characteristic. The sample is designed to best estimate "true" values or "population" values, of the characteristic. For example, if the characteristic is phosphorus concentration in a lake, there are population values for the mean and standard deviation for phosphorus concentration. A data collection program may be undertaken, and this can provide "sample" estimates of the mean and standard deviation. Techniques presented here and elsewhere (Snedecor and Cochran, 1967) may then be used to estimate the reliability of the "sample" estimates as "population" values, using, for example, the standard error of the mean.

What does sampling design, under an objective like that stated above, do for water quality assessment? Three advantages are:

1. Sampling becomes more efficient. Samples should be gathered only if they lead to a reduction in uncertainty. Thus, characteristics should not be sampled if the uncertainty in the estimates of these characteristics cannot be reduced further given the budget constraint. In addition, efficient sampling means that trade-offs should be made among different sampling schemes to achieve the most cost-effective design.

2. Sampling design stimulates thinking about the issue of concern, and it leads to a good definition of informational needs.

3. Sampling design leads to an explicit quantification of the uncertainty in the estimate of the characteristic(s) sampled.

It is shown in Chapters 3 and 4 that an estimate of uncertainty is quite valuable to both modelers and planners, as it quantifies the value of the information contained in the gathered data.

### 2.3.2 Sampling Design Relationships

It has already been stated that sampling design is a function of uncertainty (precision) and cost. In addition to those two factors, though, sampling design depends upon the variance (or the magnitude and variability) of a characteristic. For example, if the characteristic to be sampled is the phosphorus concentration in a lake, then the primary factors needed for the design of a sampling program are the cost of sampling, the precision expected from each proposed design, and an estimate of the variance in phosphorus concentration. The variance is a population statistic, and it exists largely because of natural fluctuations, or variations, in concentration throughout a lake.

Sampling design relationships are often presented on the basis of random design. Most water quality sampling, however, is systematic (e.g., samples are taken every two weeks, or at the top, middle, and bottom). However, random sampling design equations may be used for systematic sampling if there is no bias introduced by incomplete design, and if there is no periodic variation in the characteristic measured. Use is further justified if the systematic sampling begins with a random "start" (e.g., begins on a randomly selected day or at a randomly selected site or depth, with systematic sampling from that point on).

Random sampling is characterized by the following equation:

$$n = \frac{t^2 s^2}{d^2} \qquad (2.10)$$

where:

$n$ = number of samples

$t$ = student's t

$s^2$ = population variance estimate

$d$ = precision, or uncertainty, of sampling design.

This equation states that the number of samples taken should be increased if greater precision is desired, or if the population variance becomes larger. Sample number is related to cost, perhaps as:

$$C(n) = c_0 + c_1 n \qquad (2.11)$$

where cost of sampling involves a fixed cost plus a variable cost dependent upon the number of samples taken. Thus, for different sampling programs, a plot of sampling cost versus uncertainty (precision) can be constructed (Figure 2.6). "Optimal" sampling programs are those lying on the curve, as these programs minimize uncertainty for a given cost (or vice versa).

For many types of problems, sampling can be more efficient when the design is based on the fact that a population often contains strata with characteristics of different magnitudes and variabilities. For example, stratified lakes often have different conditions in the hypolimnion and epilimnion. In addition, the nutrient flux to a

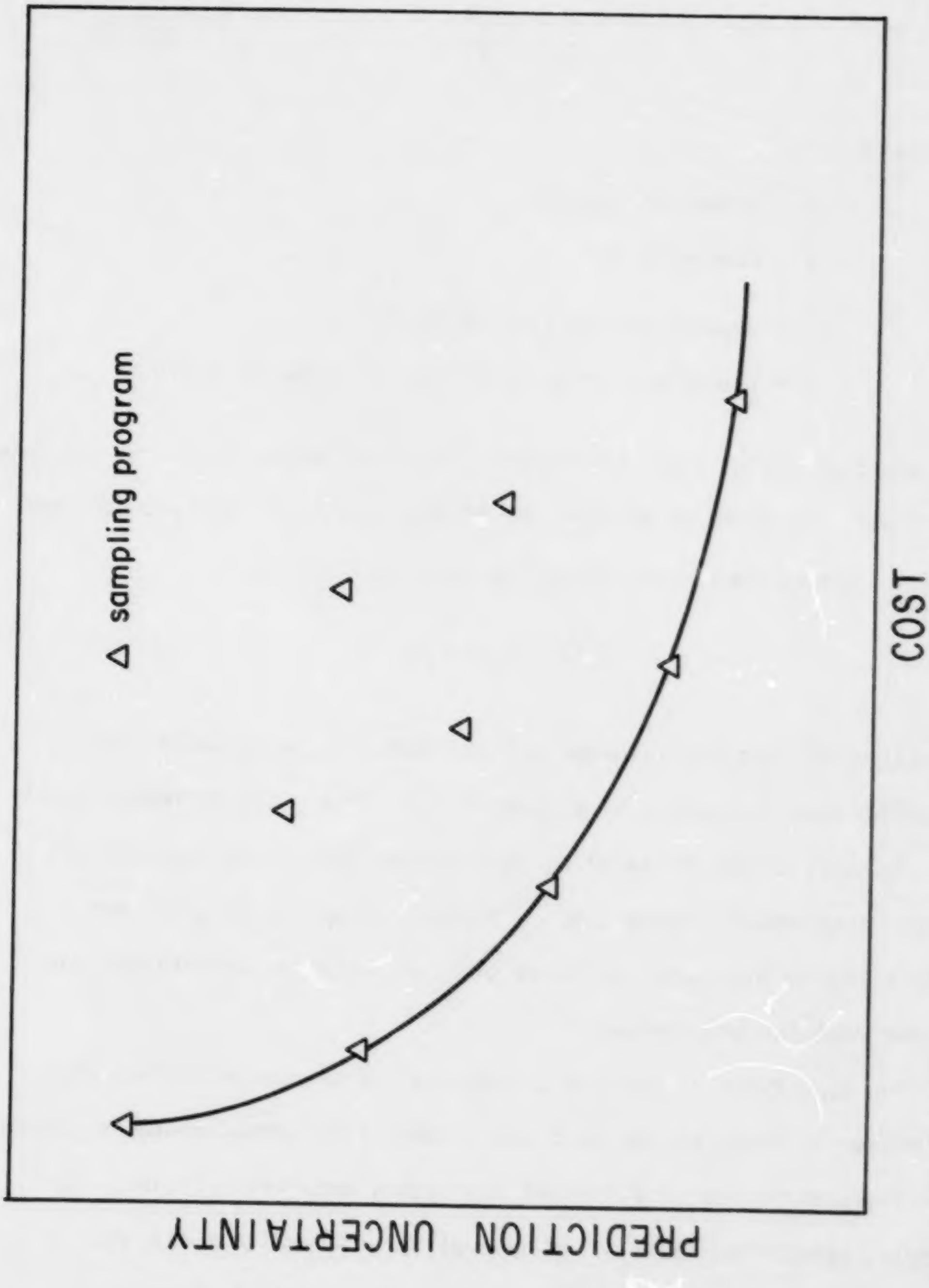

Figure 2.6 Sampling Design Uncertainty Versus Cost

lake can vary significantly from tributary to tributary. In these situations, sampling is more efficient when sample numbers are allocated according to stratified random sampling design. Then, within each stratum, sampling is random or systematic with a random start. Sampling is allocated in stratified random sampling design according to:

$$\frac{n_i}{n} = \frac{w_i(c.v.)_{x_i}(x_i)}{\Sigma w_i(c.v.)_{x_i}(x_i)} \qquad (2.12)$$

where:

$n$ = total number of samples

$n_i$ = number of samples in stratum i

$x_i$ = magnitude (mean) of characteristic x in stratum i

$w_i$ = a weight reflecting the size (number of units, for example) of stratum i

$(c.v.)_{x_i}$ = coefficient of variation (standard deviation divided by the mean) of characteristic x in stratum i.

If sampling cost may be estimated by:

$$C = c_o + \Sigma c_i n_i \qquad (2.13)$$

then

$$\frac{n_i}{n} = \frac{w_i(c.v.)_{x_i}(x_i)/\sqrt{c_i}}{\Sigma w_i(c.v.)_{x_i}(x_i)/\sqrt{c_i}} \qquad (2.14)$$

In order to apply Equation 2.12 or 2.14, a relationship is needed for the total number of samples n. Two equations are available, depending upon whether precision cost is fixed beforehand. If precision is fixed (at d), and cost may be estimated according to Equation 2.13, then (Cochran, 1963):

$$n = \frac{[\Sigma w_i (c.v.)_{x_i}(x_i)\sqrt{c_i}]\Sigma w_i (c.v.)_{x_i}(x_i)/\sqrt{c_i}}{d^2/t^2} \qquad (2.15)$$

If cost is fixed, then (Cochran, 1963):

$$n = \frac{(c - c_0)\Sigma(w_i (c.v.)_{x_i}(x_i)/\sqrt{c_i})}{\Sigma(w_i (c.v.)_{x_i}(x_i)\sqrt{c_i})} \qquad (2.16)$$

As an example, Table 2.4 presents figures for a hypothetical lake (summer conditions). These figures are based on limited existing information or informed judgment. The problem is to design a sampling program to estimate the mean phosphorus concentration to within ±.005 mg/l at the 95% level. Assume that costs of epilimnion and hypolimnion sampling are equivalent.

TABLE 2.4

ESTIMATED CONDITIONS IN A HYPOTHETICAL LAKE

| Characteristic | Epilimnion | Hypolimnion |
|---|---|---|
| mean phosphorus concentration, $x_i$ (mg/l) | .020 | .070 |
| $s_{x_i}$ (mg/l) | .010 | .050 |
| $(c.v.)_{x_i}$ | .50 | .71 |
| volume ($m^3$) | $5 \times 10^6$ | $1 \times 10^6$ |

To solve this problem, one must first note that epilimnion and hypolimnion volumes are different. Thus, one must "weight" the epilimnion concentration by 5/6 and the hypolimnion volume by 1/6, since the epilimnion volume is five times the hypolimnion volume. Therefore, the total number of samples to be taken are (using Equation 2.15):

$$n = \frac{[(5/6)(.50)(.020) + (1/6)(.71)(.070)]^2}{(.005)^2/2^2}$$

$$n = \frac{2.76 \times 10^{-4}}{6.25 \times 10^{-6}} = 44.18$$

$$n \simeq 45 \text{ samples}$$

Equation 2.12 may be used to allocate the samples between the hypolimnion and the epilimnion:

$$\frac{n_e}{n} = \frac{(5/6)(.50)(.020)}{(5/6)(.50)(.020) + (1/6)(.71)(.070)}$$

$$\frac{n_e}{n} = .5$$

where $n_e$ = number of samples to be taken in the epilimnion.

Thus, 50% of the samples (22-23 samples) are to be taken in the epilimnion, and 50% of the samples are to be taken in the hypolimnion. As a check, the precision (standard error of the mean) of this sampling design is estimated by (Equation 2.3):

$$s_{\bar{x}} = \frac{s}{\sqrt{n}}$$

For the epilimnion:

$$s_e = \frac{.01}{\sqrt{23}} = .0021 \text{ mg/l}$$

For the hypolimnion:

$$s_h = \frac{.05}{\sqrt{23}} = .0104 \text{ mg/l}$$

For the entire lake:

$$s_{\bar{x}} = \sqrt{[(5/6)(.0021)]^2 + [(1/6)(.0104)]^2}$$

$$= \sqrt{3.06 \times 10^{-6} + 3.06 \times 10^{-6}}$$

$$s_{\bar{x}} = \quad .0025 \text{ mg/l}$$

At the 95% level, $t \approx 2$, and precision, d, is:

$$d = ts_{\bar{x}}$$

$$d = .005 \text{ mg/l},$$

which corresponds to the desired precision.

In summary, the composition of the stratified random sampling design equations leads to the following general conclusions concerning stratified sampling. A larger sample should be taken in a stratum if the stratum is:

1. more variable (c.v.)
2. larger (w, x)
3. less costly to sample (c)

### 2.3.3 Estimation of Factors Important in Sampling Design

At this point, it should be apparent that before the sampling design relations can be applied, estimates are needed for the magnitude and variability, or variance, of the characteristic(s) sampled. These estimates may be, and often are, "rough," but as long as they are representative and thoughtfully selected, they will be useful in guiding sampling design, using the equations above. If one is concerned that these estimates are highly uncertain, it might be useful to calculate designs using a range of these "prior" estimates, in order to test the "sensitivity" of the design to the estimated values.

Cochran (1963) identifies four approaches for the prior estimation of magnitude and variability, or variance:

1. Use existing information on the same "population" or a similar population (e.g., existing data on lake concentration, or nutrient inflows).

2. Use informed judgment, or an educated guess.

3. Take the sample in two steps. Use the first-step sampling results to estimate the needed factors for best design of the second step. Use data from both steps to estimate the final precision of the characteristic(s) sampled.

4. Perform a "pilot study" on a "convenient" or "meaningful" subsample, and use the results to estimate the needed factors. Here the results of the pilot study generally cannot be used in the calculation of the final precision, because the pilot sample often is not representative of the entire population to be sampled. This possible non-representativeness must also be taken into account when the pilot survey results are used to estimate magnitude and variability, or variance. A modification might be necessary if it is thought that the pilot survey provided an overestimate or underestimate of the population values.

### 2.3.4 Some Specific Issues

Experience in limnology and data analysis is invaluable in the design and interpretation of sampling programs. Specifically, this experience may be useful in making adjustments to the design and analysis so that the results best represent the characteristics of interest. Many situations may benefit from this experience; a few examples are presented below.

Some lakes are irregularly shaped or contain concentration gradients due to major point sources. In those cases, stratified sampling may be undertaken with additional strata in the horizontal direction.

In addition, nutrient concentrations and nutrient flux estimates are dependent on hydrology. Severe storms can change lake nutrient concentrations and can introduce a significant fraction of the total annual nutrient loading to a lake. Unusual hydrologic years (very "wet" or very "dry" years) can also significantly affect nutrient loading and lake quality. Thus, conditions observed during the period-of-sampling may not generally be representative of conditions over time or of "average" conditions. Hydrology (and, of course, land use changes) must be taken into account.

### 2.3.5 Annual Phosphorus Budget

Sampling design can be a particularly useful exercise in the measurement of nutrient flux to a lake. Application of the sampling design equations, or of the concepts important in sampling design, can lead to efficient sampling programs based on explicit trade-offs among different sampling schemes. In addition, an estimate of the uncertainty associated with gathered data on nutrient flux is valuable information for use in the models and classification schemes in Chapters 3 and 4.

Lake phosphorus budget sampling design is discussed in considerable detail by Reckhow (1978c). This section will briefly summarize some of the issues presented in that paper. Major sources of phosphorus considered were tributaries, sewage treatment plants, urban runoff, precipitation, septic tanks - groundwater, and lake sediments. For each source, the sampling design was based on an estimation technique, or model, that converted the gathered data to an annual phosphorus flux estimate.

Phosphorus flux in lake tributaries has been studied extensively, and thus there is a substantial quantity of literature that may be used for the estimation of the expected magnitude and variability of that flux. The EPA National Eutrophication Survey is a good source of data, and many of the EPA-NES streams have been classified by land use (Omernik, 1977). In general, phosphorus concentration (in streams) decreases with flow in streams impacted by a sizeable point source, and increases with flow in streams undisturbed by major point sources. On that basis, phosphorus flux is probably best estimated by multiplying average flow times the flow-weighted concentration or by a regression equation of flux on flow. Since those calculations of flux require information on flow, it is recommended that continuous flow measurements be made, or that a regression equation (of flow on precipitation and watershed characteristics) be used to provide flow data. Regression equations like that described are available from the U.S. Geological Survey. Sampling for concentration should be allocated among tributaries using stratified random sampling, and it should probably occur on 2-4 week intervals (with a random start, and allocated according to seasonal flow variations). More frequent sampling results in auto-correlation among samples, and less frequent sampling may result in considerable error. Finally, some consideration should be given to sampling major storm events, as a large percentage of the phosphorus loading may occur during those times.

Much data also exist on wastewater treatment plants, and again the EPA-NES is a good source. Treatment plant data exhibit a distinct diurnal cycle, so composite sampling is preferrable. Phosphorus flux estimates may be made from flow-weighted concentration times flow (continuous flow data should be available). Existing EPA-NES data indicate that the average phosphorus concentration varies considerably from plant to plant, while the coefficient of variation (the variability) of phosphorus concentration generally lies between .3 and .5. Sampling among plants should be based on stratified random design, while sampling over time should be based on randomsampling to reach a desired or minimum precision.

Urban runoff sampling clearly must be geared to storm events. Insufficient data exist to guide sampling designs in most situations. Therefore, only some general recommendations can be made. Automatic sampling may be most effective, since human response to a storm may miss a portion of the "first flush." Composite sampling for concentration may be used to estimate flux, as average flow times average concentration. Grab sampling can be used to fit an exponentially-decaying concentration model (Marsalek, 1975), that may be used to estimate flux with continuous flow data.

Existing data on phosphorus in bulk precipitation (precipitation plus dry fallout) indicate considerable variability from year-to-year, site-to-site, and storm-to-storm. Bulk precipitation phosphorus results from industrial air pollution, bare agricultural fields, dirt roads, etc. In many lakes, precipitation is a relatively minor source of phosphorus. Thus, literature values for precipitation phosphorus (Uttormark, et al., 1974) should probably be compared to the expected flux of phosphorus (to the lake of study) from other sources before a sampling program is undertaken for this source.

No satisfactory techniques have yet been developed to measure phosphorus flux to a lake from septic tanks and groundwater. The most common technique used is a soil retention coefficient, specific to a soil type. However, a constant soil retention does not consider the time-dependency of retention, the total volume of soil through which phosphorus in solution must pass, and the loading of phosphorus to the soil. Probably a better technique at this time is a system of "seepage meters" in the shallow lake sediments and wells immediately onshore (Lee, 1977). The seepage meters are used to measure groundwater flow (assumed to decrease exponentially with distance from shore), and the wells are used to measure phosphorus concentration. Unfortunately, insufficient data exist to design this program, but the concepts of stratified random sampling (magnitude, variability, and cost) suggest that sampling units should be most dense in areas with the greatest density of septic tanks and in areas with soils of lowest retention coefficients.

Finally, the lake sediments are another source of phosphorus that is not well-defined. As a rule, the sediments are considered to be a significant source only under anaerobic conditions. However, studies indicate (Snow and DiGiano, 1976) that aerobic sediments often release phosphorus also. Estimation techniques, such as a constant daily release of phosphorus, or release proportional to the concentration gradient between the water column and the interstitial water, have been proposed (Reckhow, 1978c). Experimental procedures have been developed for both the laboratory and the field (Snow and DiGiano, 1976). It is suggested that "typical" release rates, presented in Reckhow (1978c) and Snow and DiGiano (1976), be compared to expected phosphorus flux from other sources, before a sampling program is undertaken for the lake sediments.

As an example of lake phosphorus budget sampling design, the following analysis was conducted to guide the sampling of phosphorus flux to Lake Winnipesaukee in New Hampshire (Reckhow and Rice, 1975; Reckhow, 1978c). This analysis emphasized the concepts of stratified random sampling (take a larger sample if more variable, larger, or costs less); it did not consist of the explicit trade-offs and computations that might be possible with the material presented above. Nonetheless, it does show, in a general sense, how sampling design may develop.

Table 2.5 presents the mean, standard deviation, and precision of existing estimates for phosphorus flux from the tributaries and the wastewater treatment plants. Informed judgment yielded the magnitude and range estimates for the other three phosphorus sources; data were deemed insufficient to specify these terms more precisely. This table then provided the basis for general sampling design recommendations, summarized in the statements below:

1. Existing estimates of the phosphorus flux from tributaries and sewage treatment plants may be sufficient (i.e., no additional sampling necessary), if they were obtained with an unbiased sampling design, and if significant changes (land use, etc.) have not occurred.

2. Considerable sampling effort should be devoted to estimating the mean and variance in phosphorus flux from septic tanks.

3. The other sources of phosphorus (sediments and precipitation) should be investigated through the literature, but they may not require sampling.

4. If tributary sampling is undertaken:
   a) spatial coverage should be based on stratified random

## TABLE 2.5

## INITIAL UNCERTAINTY ESTIMATES FOR LAKE WINNIPESAUKEE PHOSPHORUS LOADING

| | Prior Estimates | | | |
|---|---|---|---|---|
| Term | Magnitude | Coefficient of Variation | Standard Error of the Mean (%) | Estimated Range |
| 1. Tributary Flux | 16,000 lb P/yr | .65 | ±20 | |
| 2. Septic Tanks | | | | 4,000-30,000 lb P/yr |
| 3. Sewage Treatment Plants | 22,000 lb P/yr | .30 | ±10 | |
| 4. Precipitation | | | | 4,000-7,000 lb P/yr |
| 5. Sediment Release | | | | 700-7,000 lb P/yr |

sampling design (which may result in no sampling in the smallest streams that are not culturally impacted).

b) temporal coverage should consist of a sampling interval of 2-3 weeks, with sampling being more frequent during high runoff months and less frequent during months of low runoff.

### 2.3.6 Summary and Conclusions

In conclusion, a good sampling design requires the following information:

1. Prior knowledge of the factors that affect the characteristic(s) to be sampled (e.g., sources of phosphorus for a phosphorus budget).

2. Some knowledge of the magnitude and variability of the characteristic(s) to be sampled.

3. Pre-specified needs for the collected data. For example, phosphorus flux data may be used for a year-of-sampling estimate or for future predictions. Different designs and estimation techniques may be appropriate for each of these applications.

4. A knowledge of costs associated with sampling.

5. A model, or models, for estimation (when appropriate) that is compatible with the chosen sampling design.

# CHAPTER 3

# PHOSPHORUS MODELS

## 3.1 Introduction

The empirical phosphorus models discussed in this chapter are all based on the assumption that a lake may be treated as a black box. This means that for modeling purposes, internal lake mechanisms may be ignored, and models are developed to quantify processes at the lake interface only. These modeled processes are materials (phosphorus) input and output, and the interfaces to the lake are the land, atmosphere, and lake sediments. This chapter contains a review of the form of these black box models and an examination of proposed empirical approaches. These empirical models are compared and examined for bias, limitations, and sensitivities. Uncertainty analysis is then discussed as a measure of model prediction information. Finally, application of four of the models on a case study lake is used to underscore the importance of uncertainty when the models are used in lake management. The reader whose major concern is the application of these models may prefer to skip Section 3.2 and focus on Sections 3.3-3.5.

## 3.2 Historical and Theoretical Development: A Brief Review

Vollenweider (1969) expressed the lake mass balance on phosphorus in the following form:

$$V\frac{dp}{dt} = M - \sigma PV - QP \tag{3.1}$$

where, on an annual basis,

$P$ = lake phosphorus concentration (mg/l)

$V$ = lake volume ($10^6 m^3$)

$M$ = annual mass rate of phosphorus inflow to lake ($10^3$kg/yr)

$Q$ = annual volume rate of water outflow from lake ($10^6 m^3$/yr)

$\sigma$ = sedimentation coefficient ($yr^{-1}$).

Equation 3.1 states that the change in phosphorus mass in the lake (VdP) per unit time (dt) is equal to the mass input of phosphorus (M) minus the mass output of phosphorus through the outflow (QP) minus the net mass of phosphorus deposited in the sediments ($\sigma$PV). This sediment "sink" term for phosphorus is based on the assumption that the rate of phosphorus deposition to the sediments is proportional to the total mass of phosphorus in the lake (PV). Note that the lake and outflow concentrations of phosphorus are assumed to be equivalent. Equation 3.1 has the time-dependent solution:

$$P = \frac{L}{\sigma z + z/\tau}\left[1 - e^{-\left(\frac{1}{\tau} + \sigma\right)t}\right] + P^{\circ}\ e^{-\left(\frac{1}{\tau} + \sigma\right)t} \qquad (3.2)$$

and the steady-state solution $\left(\frac{dP}{dt} = 0\right)$:

$$P = \frac{L}{z\left(\frac{1}{\tau} + \sigma\right)} \qquad (3.3)$$

where

$L$ = annual areal phosphorus loading (g/$m^2$-yr)

$z$ = lake mean depth (m)

$\tau$ = hydraulic detention time (yr)

$t$ = averaging time length (yr)

$P^{\circ}$ = average lake phosphorus concentration at time $t = 0$ (mg/l).

The phosphorus mass balance may also be expressed as:

$$V\frac{dP}{dt} = M - v_s PA - QP \qquad (3.4)$$

where

$A$ = lake surface (bottom) area (km$^2$)

$v_s$ = apparent settling velocity (m/yr).

Equation 3.4 is different from Equation 3.1 in that the sediment sink term is treated as an areal sink. Thus, the rate of deposition of phosphorus to the sediments is a function of the bottom (surface) area. This results in the steady-state model:

$$P = \frac{L}{v_s + z/\tau} \qquad (3.5)$$

where

$z/\tau = q_s$ = surface overflow rate (m/yr).

The major difference between the two steady-state models (Equations 3.3 and 3.5) is that one model is based on the assumption of a depth-dependent settling velocity ($\sigma z$), while the other model is based on the assumption of a constant settling velocity ($v_s$).

A third steady-state phosphorus model form results from the definiton of $R_p$, the fraction of influent phosphorus retained in the lake:

$$R_p = \frac{M - QP_o}{M} \qquad (3.6)$$

where $P_o$ = average outflow phosphorus concentration (mg/l). Since M = LA and Q/A = z/τ,

$$R_p = \frac{L - (z/\tau)P}{L} = \frac{\frac{L\tau}{z} - P_o}{\frac{L\tau}{z}}$$

where

$\frac{L\tau}{z}$ = average influent phosphorus concentration (mg/l).

If it is assumed that $P_o = P$ (lake and outflow concentrations are equal), then the third model becomes:

$$P_o = P = \frac{L\tau}{z}(1 - R_p) \qquad (3.7)$$

The three basic model forms, from which the empirical models are estimated, are Equations 3.3, 3.5, and 3.7. By setting Equation 3.3 equal to Equation 3.7, it may be shown that:

$$R_p = \frac{\sigma}{1/\tau + \sigma} = \frac{1}{1 + 1/\tau\sigma} \qquad (3.8)$$

In a similar manner, the relationship between R and $v_s$ (Equations 3.5 and 3.7) is:

$$R_p = \frac{v_s}{v_s + z/\tau} = \frac{v_s}{v_s + q_s} \qquad (3.9)$$

where $q_s = z/\tau$.

To understand the appropriate criteria that must be considered in selecting the "best" model form (or, more fundamentally, to determine whether a "best model form" exists at all), we should examine the phosphorus mass balance in its most general form:

$$V\frac{dP}{dt} = M_I - M_o - \phi\,(M_I, M_o, P, Ca, Fe, Al, pH, O_2, \tau, z, z/\tau, V, \ldots) \qquad (3.10)$$

where

$M_I$ = annual mass influx of phosphorus

$M_o$ = annual mass efflux of phosphorus

$\phi$ = annual net flux of phosphorus to the sediments

The removal of phosphorus from a lake occurs through two pathways, the outlet (represented by $M_o$) and the sediments (represented by $\phi$). $\phi$ is a multidimensional variable; it is dependent upon the influent and effluent phosphorus mass, lake geomorphology and hydrology, the dissolved oxygen concentration and the pH at the sediment-water interface, major cations (Ca, Fe, and Al) that combine with phosphorus and transport it to or hold it in the sediments, and probably additional mechanisms important on a limited scale. Thus, the expression of the sediment removal term as a phosphorus retention coefficient, an apparent settling velocity, or a sedimentation coefficient, with the functional dependencies as expressed in Equations 3.1 - 3.9, is actually a simplification of the true form. Since the three basic model forms are simplified versions of reality, any discussion over the "best" form of the sediment sink term must be of an empirical, and not theoretical, nature, since $\sigma$, $v_s$, and $R_p$, as functionally defined in Equations 3.1 - 3.9, are incomplete representations of $\phi$.

## 3.3 Empirical Models

The first attempt to develop an empirical fit to one of the mass balance models presented in the previous section was by Vollenweider (1975) for Equation 3.3. From an analysis of available data (not specified further), Vollenweider found that $\sigma$ could be estimated by:

$$\sigma = \ln 5.5 - 0.85 \ln z \qquad (3.11)$$

$$(r = 0.79)$$

or

$$\sigma \approx \frac{10}{z}\,. \qquad (3.12)$$

Thus, based on Equation 3.12, Vollenweider implicitly advocated a constant settling velocity ($v_s$ = 10 m/yr) empirical model.

Dillon and Rigler (1974) looked at Equation 3.7 in a paper entitled: "A Test of a Simple Nutrient Budget Model Predicting the Phosphorus Concentration in Lake Water." However, it must be emphasized that Dillon and Rigler did not test an empirical formulation for $R_p$, but rather they examined through Equation 3.7, the frequently invoked assumption that lake and outflow concentrations are identical. It has been shown (Reckhow, 1977) that this assumption is strongly supported by available data.

Equations 3.8 and 3.9 suggest that $R_p$ may be empirically estimated as a function of $z/\tau$ or $\tau$. Kirchner and Dillon (1975) confirmed this in a model for $R_p$, developed from a data set of 14 Canadian Shield lakes:

$$R_{KD} = 0.426 \exp(-0.271\ z/\tau) + 0.574 \exp(-0.00949\ z/\tau).^*$$
$$(r = .94) \qquad (3.13)$$

This results in the "Dillon-Rigler" or "Dillon-Rigler-Kirchner" model:

$$P = \frac{L\tau}{z}(1 - R_{KD}) \qquad (3.14)$$

Dillon and Rigler (1975) proposed that the model expressed in Equation 3.14 be used to assess the impact of land use on lake quality.

Chapra (1975), in a comment on Equation 3.13, noted the relationships between $R_p$ and $\sigma$ (Equation 3.8), and $R_p$ and $v_s$ (Equation 3.9). Using the Canadian Shield lake data, Chapra found that $v_s$ = 16 m/yr provided the best fit to Equation 3.9. This model has recently been applied by Chapra (1977) to predict present and future concentrations of phosphorus in the Great Lakes. Chapra's model and the Kirchner-Dillon model are presented in Figure 3.1. Dillon and Kirchner (1975) responded to Chapra with a statement equating the $v_s$ and $\sigma$ models for P. Actually, the implicit assumption behind the differential Equations 3.1 and 3.4 is that $\sigma$ (Equations 3.1 - 3.3) and $v_s$ (Equations 3.4 and 3.5) are constants, not functions of limnological variables. Thus despite Dillon's and Kirchner's contention that the models presented in Equations 3.3 and 3.5 are equivalent, they are equivalent only if $v_s$ is re-expressed in Equation 3.4 as a linear function of z. Dillon and Kirchner concluded their reply with a revised best-fit for $v_s$ of 13.2 m/yr. This change from Chapra's estimate of 16 m/yr was proposed

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$^*\exp(a) = e^a = (2.71828....)^a$

because Dillon and Kirchner rejected data from two "outlier" lakes in Figure 3.1 based on suspected errors in data collection. It should be noted that this conclusion on data errors resulted from comparisons of observations with model predictions and not from known field or laboratory problems (so it is not immediately clear which of the two figures, 13.2 or 16, has more merit). These models may be expressed as:

$$P = \frac{L}{16 + z/\tau} \quad \text{(Chapra)} \qquad (3.15)$$

$$P = \frac{L}{13.2 + z/\tau} \quad \text{(Dillon-Kirchner)} \qquad (3.16)$$

In an essentially concurrent study, Larsen and Mercier (1975) examined data collected through the EPA National Eutrophication Survey (EPA-NES) for northeast and north central U.S. lakes. Using a data set of 20 lakes, they found the best fit for phosphorus retention to be:

$$R_{LM} = \frac{1}{1 + 1.12 \left(\frac{1}{\tau}\right)^{0.49}}, \qquad (3.17)$$

or, more simply,

$$R'_{LM} = \frac{1}{1 + \sqrt{1/\tau}}. \quad (r = .94) \qquad (3.18)$$

This results in the Larsen-Mercier model:

$$P = \frac{L\tau}{z} (1 - R_{LM}) \qquad (3.19)$$

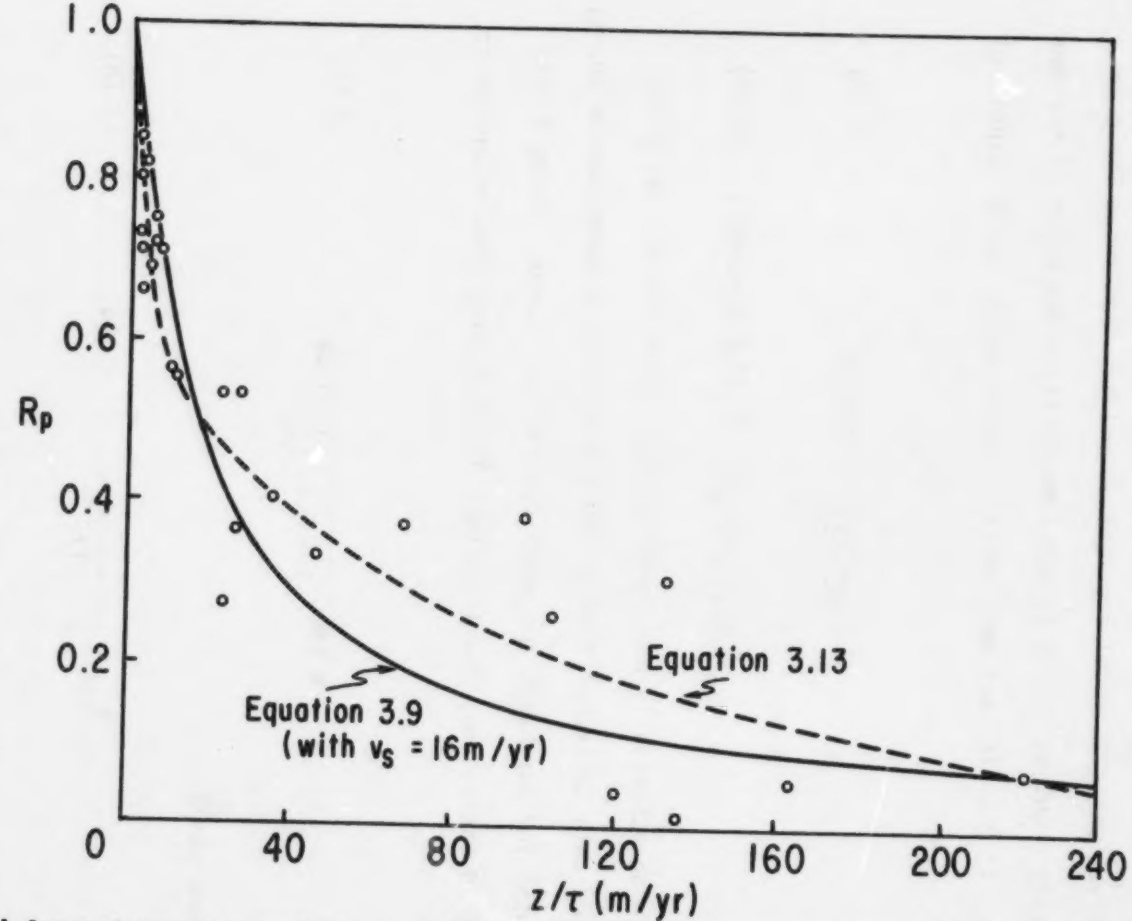

Figure 3.1 A Comparison of the Kirchner and Dillon Model For Rp With Chapra's Constant Apparent Settling Velocity Approach

Note that Equation 3.18 is identical to Equation 3.8 when $\sigma$ becomes variable and is set equal to $1/\sqrt{\tau}$.

Finally, in a recent paper, Jones and Bachmann (1976) examined data on a large number of north temperate lakes and fit Equation 3.3 with $\sigma = 0.65\ yr^{-1}$ and with a multiplicative factor of 0.84. Thus, their model is:

$$P = \frac{0.84\ L}{z\ (0.65 + 1/\tau)} \qquad (3.20)$$

Much of the previous modeling work forms the basis for comprehensive studies of the empirical models by Reckhow (1977) and Walker (1977) on north temperate lakes. Reckhow noted shortcomings in some of the empirical models, focusing on the incomplete nature of the "sediment sink" term. Because of the incompleteness of that term, it was thought that improvements in the models might result if lakes were classified according to those processes, important in sedimentation, that may not easily be modeled, or according to those variables that produce a nonlinear response in the dependent variable (total phosphorus concentration). Then, models could be developed for each of the lake classes. A model for one of these classes (oxic lakes, with $z/\tau < 50$ m/yr) is given in Table 3.1; additional statistics, discussed later in this chapter are presented in the Appendix. The remaining class models and a quasi-general model* are also presented in the Appendix. Table 3.2 presents a general model proposed by Walker, while additional model information is presented in the Appendix.

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*This model is called "quasi-general" because lakes with high phosphorus concentrations ($P > .150$ mg/l) and high qs-values ($qs > 200$ m/yr) were excluded from the analysis. The development of this model is discussed in the Appendix of Chapter 4.

## TABLE 3.1

## MODEL PROPOSED BY RECKHOW (1977)
## FOR OXIC LAKES WITH $z/\tau < 50$ m/yr

| Model | $R^{2*}$ | Standard Error of the Estimate* |
|---|---|---|
| $P = \dfrac{L}{\frac{18z}{10 + z} + 1.05\ z/\tau\ e^{.012z/\tau}}$ | .876 | .123 |

*Based on a logarithmic transformation of terms.

## TABLE 3.2

## MODEL PROPOSED BY WALKER (1977)

| Model | $R^{2**}$ | Standard Error of the Estimate** |
|---|---|---|
| $P = \frac{L\tau}{z}\left[\frac{1}{1 + .824\tau^{.454}}\right]$<br>or<br>$P = P_{in}\left[\frac{1}{1 + .824\tau^{.454}}\right]$<br>where $P_{in}$ = average influent phosphorus concentration (mg/l) | .906 | .171 |

**Based on a logarithmic transformation of all terms. These statistics compare the predicted concentration with the observed outlet concentration.

Table 3.3 presents a comparison of the empirical models discussed in this section. The evaluative criteria describe the appropriateness of each model for a given application and the value of the information provided by the model. Recall that empirical models should not be applied outside the bounds of the data set used to construct the model (without prior verification). This means, for example, that the Dillon-Rigler-Kirchner model, which was fitted on a data set of phosphorus-poor Canadian Shield lakes, was not corroborated, or verified, in the literature for use outside that geographic region. Reckhow (1977) has tested (or "verified") this model on a large data set of north temperate lakes, thus removing this geographic constraint. However, it was found that the Dillon-Rigler-Kirchner model may yield biased predictions for phosphorus-rich lakes (Reckhow, 1977). A similar, though less severe phosphorus-rich-lake bias was found for the Larsen-Mercier model (see Table 3.3).

In Table 3.3, the bounds of the data sets used to develop each model are described, in part, under the "data base" and "known constraints" columns. The "known biases" and "comments" column entries are taken from the study by Reckhow (1977). The uncertainty column indicates whether or not the modeler provided model standard error and parameter statistics so that error can be explicitly evaluated. Entries such as "unspecified," "unknown," "not evaluated," and "not considered" must be taken as indicators of the need for cautious use of the models or of the model's restricted and indeterminate value in application.

In conclusion, the differences among these models may appear subtle up to this point. It remains to be argued, in the next section,

TABLE 3.3: A COMPARISON OF EMPIRICAL MODELS

| Model | Data Base | Known Constraints* | Known Biases* | Consideration of Uncertainty | Comments |
|---|---|---|---|---|---|
| Constant $v_s$ | | | | | Based on the range of apparent settling velocities observed in natural lakes (Reckhow, 1977), these models will probably overestimate P in lakes with high z/T and underestimate P in highly enriched lakes. |
| 1) Vollenweider<br>$v_s = 10$ m/yr | unspecified | unknown | not evaluated | not considered | |
| 2) Chapra<br>$v_s = 16$ m/yr | 14 Canadian Shield lakes | $P < .015$ mg/l<br>$LT/z < .050$ mg/l | not evaluated | not considered | |
| 3) Dillon & Kirchner<br>$v_s = 13.2$ m/yr | 12 Canadian Shield lakes | $P < .015$ mg/l<br>$LT/z < .050$ mg/l | not evaluated | not considered | |
| Constant $\sigma$ | | | | | |
| Jones & Bachmann<br>$\sigma = 0.65\ yr^{-1}$ | 51 natural lakes, primarily north temperate | unknown | not evaluated | not considered | This model will probably overestimate P in shallow lakes with high values of z/T. |
| $R_p$ - regression | | | | | |
| 1) Kirchner & Dillon<br>$R_{KD}$ | 14 Canadian Shield lakes | $P < .015$ mg/l<br>$LT/z < .050$ mg/l | significant underestimation of P in lakes with $LT/z > .300$ mg/l | not considered | $R_{LM}$ was found (Reckhow, 1977) to be less biased than $R_{KD}$ for phosphorus-enriched lakes. |
| 2) Larsen & Mercier<br>$R_{LM}$ | 20 north temperate lakes, primarily oligotrophic | $P < .012$ mg/l<br>$LT/z < .025$ mg/l | significant underestimation of P in lakes with $LT/z > .300$ mg/l | not considered | |

TABLE 3.3 (continued): A COMPARISON OF EMPIRICAL MODELS

| Model | Data Base | Known Constraints* | Known Biases* | Consideration of Uncertainty | Comments |
|---|---|---|---|---|---|
| **Reckhow Models** | | | | | |
| 1) Quasi-General Model | 47 north temperate lakes | $P < .135$ mg/l<br>$LT/z < .298$ mg/l<br>$q_s < 187.$ m/yr | none significant | Parameter errors and model error are presented so that the total uncertainty may be calculated. This permits the assignment of confidence limits to a prediction. | Linked to a graphical error analysis in Chapter 4. |
| 2) Oxic, $z/T < 50$ m/yr lakes | 33 north temperate lakes | $P < .060$ mg/l<br>$LT/z < .298$ mg/l | none significant | | Smaller errors than $R_{KD}$ and $R_{LM}$ on independent data set (Reckhow, 1977). |
| 3) $z/T > 50$ m/yr lakes | 28 north temperate lakes | $P < .135$ mg/l<br>$LT/z < .178$ mg/l<br>$z < 13$ m<br>$T < 0.25$ yr | none significant | | Verification not conclusive (Reckhow, 1977). |
| 4) Anoxic lakes | 21 north temperate lakes | $.017$ mg/l $< P < .610$ mg/l<br>$.024$ mg/l $< LT/z < .621$ mg/l | none significant | | Needs to be verified. |
| **Walker Model** | 105 north temperate lakes | similar to those in Table 3A.2 | not evaluated | Prediction confidence limits can be estimated. | No major shortcomings known. |

*Constraints or biases that may significantly affect the value of the model for prediction purposes.

that the selection of the best model for use (given the present state of the art) is one that incorporates explicit consideration of uncertainty. Aside from that, though, the choice of the appropriate model from a set of simple, empirical models should be based on:

1. The data base used to fit the model.
    a. Does it contain lakes similar to yours (similar in terms of geography, climate, size, depth, trophic state, etc.)?
    b. Is the data base large (lending more credibility to the estimates of the model parameters)?

2. Previous experience with the model. Has the model been used successfully by you and others on similar lakes?

3. Model Documentation. Are model use, misuse, and limitations carefully outlined? The manual by Dillon and Rigler (1975) is a good first cut at this, but it is somewhat weak on the misuse and limitations. This is particularly important if you have had little prior experience with these models.

### 3.4 Error Analysis

It has been stated earlier in this manual that quantification of all uncertainty associated with the application of a model can provide the decision maker with information that may be used to weigh the value of model output. This section discusses the use of first-order uncertainty analysis (Cornell, 1972) for the estimation of the confidence limits associated with the prediction of one of the empirical models. Error analysis, like statistical sampling design (Chapter 2), should become a standard or routine step in comprehensive water quality management. Detailed development of the relationships used in error analysis is given in the Appendix to Chapter 3. In this section, the importance of error or uncertainty in decision making is discussed and illustrated through example.

Uncertainty is present in most water quality assessments, simply because we cannot sample throughout time and space. Uncertainty is also present in all water quality predictions, because predictions imply the use of a model. Since a model is a simplification of reality, it is imprecise. Clearly then, predictive information is more valuable if we know how precise it is. Yet, model prediction uncertainty is generally not part of model documentation. As a result, we have no information on the prediction uncertainty associated with the application of most of the models discussed in the last section.

At present, only the models developed by Reckhow and the model proposed by Walker are accompanied by the statistics needed to estimate prediction uncertainty. The prediction uncertainty estimated with these models may then be used to weigh the value of the model prediction; the smaller the prediction uncertainty, the more precise the prediction, and the greater the weight that may then be associated with the prediction. This is useful information for a decision maker. The output from the model must be considered as one contribution to the lake management process. If the modeler can also provide the decision maker with a quantitative assessment of the value of the model output in the form of prediction uncertainty, then the decision maker will be better able to use the model prediction wisely.

The application of uncertainty analysis may also be viewed in the following way. Uncertainty analysis provides confidence limits, or an estimate of the level of statistical significance at which we can discriminate the expected difference in outcomes of two strategies. In other words, the model predictions for different phosphorus loadings (different strategies) have confidence limits associated with them, calculated from the uncertainty analysis. The extent of overlap of these confidence limits determines the level of statistical significance. The statistical significance level, in turn, quantifies our ability to state that two strategies will have different outcomes (discrimination). This analysis is done like a t-test, as described in Chapter 2, with an infinite number of degrees of freedom.

So that the estimation and application of uncertainty are understood, an example is provided below. The model presented in Table 3.1 is used to predict the phosphorus concentration in Lake Ontario and to assign confidence limits to this prediction. Specifically, this approach is used to examine the likelihood of violation of a hypothetical phosphorus standard for Lake Ontario. While few standards are presently stated in a probabilistic form, the models and the phosphorus loading estimates are well suited for probabilistic statements. The expression of water quality standards in probabilistic terms is more realistic for both mathematical models and conceptual models, since both reflect uncertainty in the behavior of a lake system. Given uncertainty in the applied model and in the independent variables, the probability of meeting a lake quality standard (for phosphorus) can be calculated. This is done in Figure 3.2 for Lake Ontario ($z$ = 89 m,

$z/\tau$ = 11.3 m/yr), assuming a total phosphorus standard of 0.045 mg/l, and assuming the independent variable error exists in the phosphorus loading estimate only. (The large volume should dampen out hydrologic budget fluctuations.) Curves are presented in Figure 3.2 covering a range of loading levels.

Let us now discuss how Figure 3.2 was developed and how it may be interpreted. Total prediction error may be estimated from Equation 3.21 given below, under the following general conditions:

$$s_T'^2 = \left[ P_e - 10^{\log P_e \pm (s_p^2 + s_m^2)^{1/2}} \right]^2 + s_{iv}'^2 \qquad (3.21)$$

where

$s'_T$ = total prediction error (mg/l)

$P_e$ = predicted phosphorus concentration (mg/l)

$s_p$ = parameter error contribution to the total error (logarithmic form)

$s_m$ = standard error of the estimate for the model (logarithmic form)

$s'_{iv}$ = contribution to the error from the independent variables, primarily phosphorus loading (mg/l)

First, estimates must be available for the parameter error and model error terms. These are presented in Table 3.1, Table 3.2, and in the Appendix, for the Reckhow and Walker models. Note that they are given for the log-transformed model (for direct insertion into Equation 3.21), since these models were constructed under a logarithmic transformation. Second, an estimate is needed for the independent

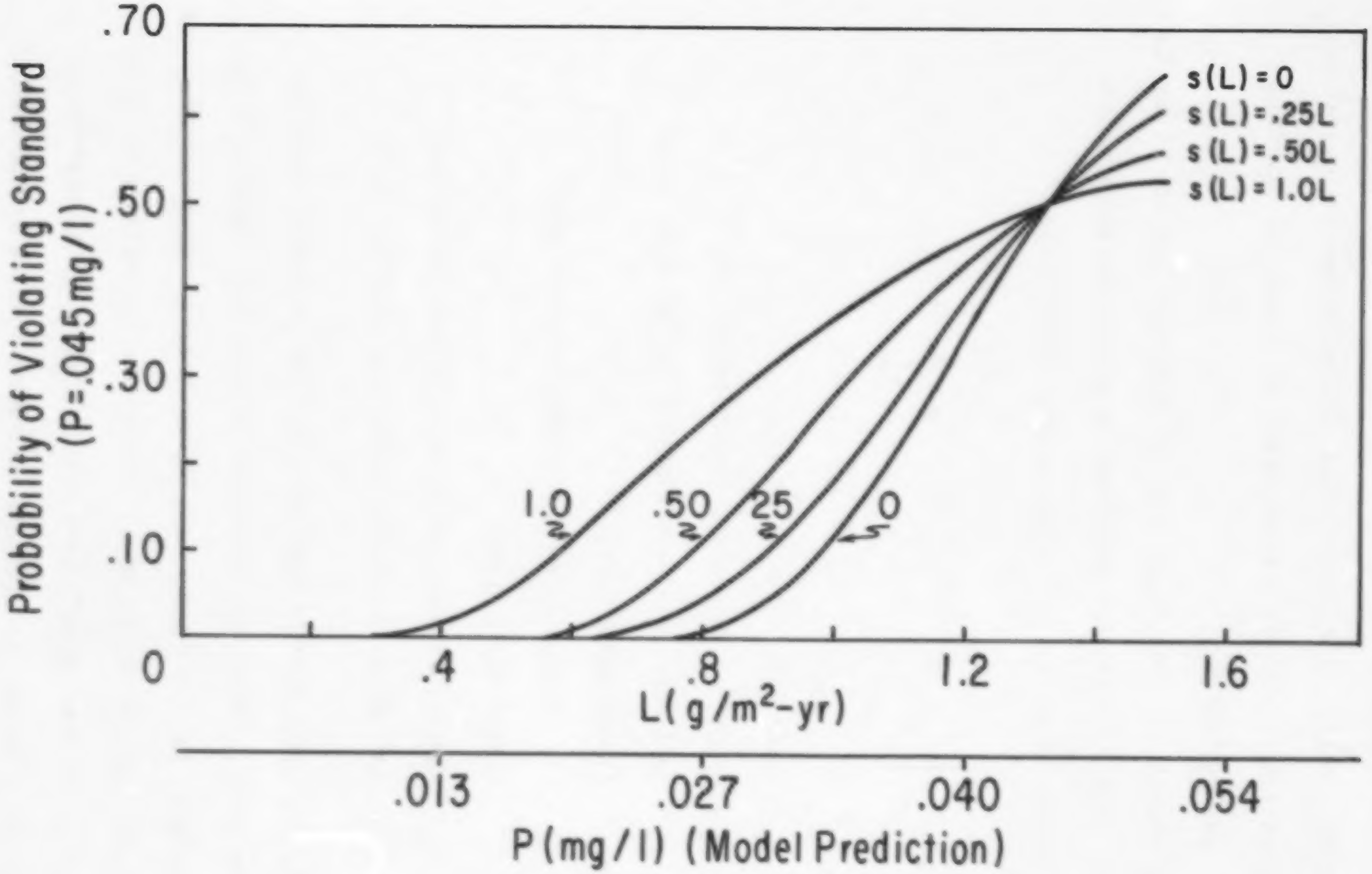

Figure 3.2 Lake Ontario: Probability of Standard (P=.045 mg/l) Violation

variable (L, z, z/τ, or τ) error contribution. For most lakes (except perhaps those with low values of τ), this error contribution is a function of the phosphorus loading only.* It is assumed here that the independent variable error is a function of the phosphorus loading only, and that it is proportional to the loading. This means that the independent variable error contribution (to the total error) is proportional to the phosphorus concentration (since the model is linear in L). This can be written as:

$$s'_{iv} = kP_e \qquad (3.22a)$$

or

$$s'_{iv} = \frac{1}{B}\,(s(L)) = \frac{1}{B}\,(kL) \qquad (3.22b)$$

where B is defined in Table 3A.6 (the denominator of the Reckhow model), and s(L) is the phosphorus loading error estimate.**

With theseterm definitions and the error terms for the Reckhow model from Table 3.1 and the Appendix (the equation for $s_p$ is solved using Lake Ontario data for z and z/τ; the result is $s_p = 4.02 \times 10^{-2}$), Equation 3.23 may be written and used to estimate the total error in milligrams per liter:

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*When this is not true, the error propagation equation, presented in the Chapter 3 Appendix, may be used to estimate the independent variable error contribution from all varibles. When this is true, the source and quality of the phosphorus loading data determines how loading uncertainty is combined with model and parameter uncertainty; see Section 3.5.

**It is important that the units be consistent. $s'^2_{iv}$ is placed where it is in Equation 3.21 because it is expressed in units of ± mg/l. If instead, this error contribution term is expressed as a log transform, then Equation 3.21 should be rewritten so that $s^2_{iv}$ is added directly to $s^2_m$ and $s^2_p$ in the exponent.

$$s_{\acute{T}}^2 = \left[P_e - 10^{\log P_e} \pm [(4.02 \times 10^{-2})^2 + (.123)^2]^{\frac{1}{2}}\right]^2 + (kP_e)^2 \qquad (3.23)$$

Then, assuming normally distributed errors, a phosphorus standard of .045 mg/l, and a predicted phosphorus concentration (from the model) of $P_e$, the probability of a standard violation may be estimated from the value of $Z_n$ (the standard normal deviate) and a cumulative normal frequency distribution table:

$$Z_n = \frac{.045 - P_e}{s_{\acute{T}}}$$

The curves in Equation 3.23 represent different values of "k" (expressing the fractional loading error); the curve marked 0 indicates the effect of model and parameter error alone. In order to use this graph, we must decide: 1) the likelihood of standard violation we are willing to accept, and 2) the uncertainty associated with our estimate of the phosphorus loading, L. Assume that we want a 10% likelihood of violation of this standard, and that we think we can estimate future* phosphorus loading to within ± 50% of its true value. Then, we look across the graph, from .10 on the vertical axis, to the .50 curve, and down to the L-axis. This corresponds to about .79 $g/m^2$-yr on the L-axis and about .0265 mg/l on the P-axis (the predicted concentration, using the Reckhow oxic model from Table 3.1). Thus, to achieve a

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*Future phosphorus loading predictions require the additional uncertainty term, $s_{\acute{l}_V}$, in Equation 3.21. An explanation for this is provided in Section 3.5.

90% likelihood of standard compliance (10% likelihood of violation), when the future loading can be estimated to within ± 50%, we recommend a design loading of about .79 g/m$^2$-yr, which has an expected phosphorus concentration of about .0265 mg/l. The difference between the standard, .045 mg/l, and the design concentration, .0265 mg/l, may be considered an "engineering safety factor." It exists because we have uncertainty in the model and in the loading estimate, and because we desire a small likelihood of standard violation. Note that if we design for a 50% likelihood of standard violation, then the design concentration equals the standard concentration.

This graphical approach has been used by Reckhow and Rice (1976, 1977) in the evaluation of lake management plans for a "208" Program in New Hampshire. This method facilitated the understanding of uncertainty, and it permitted the comparison of alternative strategies. Most important, in this investigator's view, it is "honest" in that it clearly indicates that, in fact, we cannot be certain that a strategy will or will not result in a standard violation.

### 3.5 A Case Study Comparison of Four Models

As further illustration of the usefulness of error analysis, four models (Dillon-Rigler-Kirchner, Larsen-Mercier, Walker, and Reckhow oxic) are applied below to estimate the total phosphorus concentration in Lake Charlevoix in Michigan. This lake was randomly selected from a group of oligotrophic lakes from the EPA National Eurtophication Survey (EPA-NES). Data reported by the EPA-NES (1975) are given in Table 3.4.

TABLE 3.4

DATA FOR LAKE CHARLEVOIX
(EPA-NES, 1975)

| | |
|---|---|
| $z = 16.76$ m | $L = 0.12$ g/m$^2$-yr |
| $\tau = 3.2$ yrs | $P_{in} = L\tau/z = .02291$ mg/l |
| $z/\tau = q_s = 5.238$ m/yr | $P = .007$ mg/l<br>(Fall mean and median) |

The Dillon-Rigler-Kirchner model is (Equation 3.14):

$$P = \frac{L\tau}{z}(1 - R_{KD})$$

where:

$$R_{KD} = 0.426 \exp(-0.271\, z/\tau) + 0.574 \exp(-0.00949\, z/\tau)$$

Substituting:

$$R_{KD} = 0.426 \exp[-0.271\,(5.238)] + 0.574 \exp[-0.00949\,(5.238)]$$

$$= .1030 + .5462$$

$$= .6492$$

Thus:

$$P = \frac{L\tau}{z}(1 - R_{KD})$$

$$= .02291\,(1 - .6492)$$

$$= .0080 \text{ mg/l}$$

The Larsen-Mercier model is (Equation 3.19):

$$P = \frac{L\tau}{z}(1 - R_{LM})$$

where:

$$R_{LM} = \frac{1}{1 + 1.12\left(\frac{1}{\tau}\right)^{0.49}}$$

Substituting:

$$R_{LM} = \frac{1}{1 + 1.12\left(\frac{1}{3.2}\right)^{0.49}}$$

$$= .6122$$

Thus:

$$P = \frac{L\tau}{z}(1 - R_{LM})$$

$$= .02291\ (1 - .6122)$$

$$= .0089\ \text{mg/l.}$$

Before the uncertainty analysis is applied in conjunction with the Walker and Reckhow models, some discussion on the appropriateness of the terms in Equation 3.21 for prediction error estimation is in order. The error analysis procedure based on Equation 3.21 is most appropriate when the error in the variables in the data set used to develop the model contributes a small or negligible amount to the total prediction error. However, estimates of phosphorus loading, in particular, can be quite uncertain, as can the estimates of average phosphorus concentration and hydraulic detention time for selected lakes. Thus, it is likely that a non-negligible fraction of the model standard error results from the errors in these variables (in the model development data set). In that case, two recommendations are made, given the existing models and model statistics.

1. If a model is used to assess current conditions in a lake for which the phosphorus input and hydraulic detention time are direct measurements, the level of uncertainty in these variables probably is approximately the same as the average uncertainty in these terms in the model development data set (for the models considered here). In that case, the uncertainty in these variables is incorporated in the model standard error, and the prediction confidence limits may be estimated using Equation 3.21 with $s^2_{i.v.} = 0$ ($k = 0$ in Equation 3.23 and on Figure 3.2).

2. If a model is applied to a lake for which the independent variables, particularly phosphorus loading, are indirect estimates (e.g., literature export coefficients for phosphorus loading, or direct measurements for future predictions), then it is likely that the uncertainty in these estimates is quite large. The component of the model standard error contributed by the errors in the variables is probably small in comparison. In this case, the best estimate of the total prediction uncertainty would be provided by Equation 3.21, unmodified.

Therefore, to illustrate the use of the Walker and Reckhow models, calculations are presented under both conditions 1 and 2 above. Under condition 1, the EPA-NES data in Table 3.4 may be used in an error analysis with $k=0$. Then, for the purpose of example, we can use the same EPA-NES data under condition 2. That is, we will assume that the EPA-NES loading data has an uncertainty of ± 50% for future predictions, and then we can calculate the resultant prediction confidence limits using Equation 3.21.

The Walker model may be applied using Table 3.2, Equation 3.21, and Equation 3.22a. The model is (Table 3.2):

$$P = \frac{L\tau}{z}\left[\frac{1}{1 + .824\tau^{.454}}\right]$$

Substituting:

$$P = .02291\left[\frac{1}{1 + .824(3.2)^{.454}}\right]$$

$$= .02291\ (.4171)$$

$$= .0096\ \text{mg/l}$$

The parameter error contribution may be calculated from (Table 3A.5):

$$s_p^2 = 0.001\ Y^4\ \tau^{.908}\ [4.49 + 1.44\ (\ln\ \tau)^2 + 0.032\ \ln\ \tau)]$$

where:

$$Y = 1/(1 + .824\tau^{.454})$$

$$= .4171$$

Substituting:

$$s_p^2 = 0.001\ (.4171)^4\ (3.2)^{.908}\ [4.49 + 1.44\ (\ln\ 3.2)^2 + 0.032\ \ln\ (3.2)]$$

$$= 0.001\ (3.027 \times 10^{-2})\ (2.875)\ (6.475)$$

$$= 5.635 \times 10^{-4}$$

The model error is presented in Table 3.2:

$$s_m = .171.$$

Now, Equation 3.21, with $s_{iv}' = 0$, may be used to estimate the prediction uncertainty:

$$s_T'^2 = \left[P_e - 10^{\log} P_e \pm [5.635 \times 10^{-4} + (.171)^2]^{\frac{1}{2}}\right]^2$$

After substituting the predicted concentration ($P_e$ = .0096 mg/l) into the above relation, we can estimate the confidence limits*:

$$s_T'^2 = \left[.0096 - 10^{\log} (.0096) \pm .173\right]^2$$

$$s_T' = + .0047 \text{ mg/l}$$

$$s_T' = - .0032 \text{ mg/l}$$

This result may be expressed as:

$$.0064 \text{ mg/l} < P_{true} < .0143 \text{ mg/l*}$$

where $P_{true}$ = the "true" total phosphorus concentration (mg/l).

For condition 2, the independent variable error term may be estimated from Equation 3.22a:

$$s_{iv}' = .50\ P_e$$

---

*The confidence limits given here equal ± one standard error. This means that about 68% of the time that the standard error is estimated, the "true" value falls within ± one standard error of the predicted value.

The three error components may now be substituted into Equation 3.21, to estimate the prediction uncertainty:

$$s_{\acute{T}}^2 = \left[P_e - 10^{\log P_e} \pm [5.635 \times 10^{-4} + (.171)^2]^{\frac{1}{2}}\right]^2 + (.50\, P_e)^2$$

After the predicted concentration ($P_e = .0096$ mg/l) is substituted into the above relation, the confidence limits* can be calculated:

$$s_{\acute{T}}^2 = \left[.0096 - 10^{\log(.0096) + .173}\right]^2 + [(.50)(.0096)]^2$$

$$= (.0047)^2 + (.0048)^2$$

$$s_{\acute{T}} = + .0067 \text{ mg/l}$$

and

$$s_{\acute{T}}^2 = \left[.0096 - 10^{\log(.0096) - .173}\right]^2 + [(.50)(.0096)]^2$$

$$= (.0032)^2 + (.0048)^2$$

$$s_{\acute{T}} = - .0058 \text{ mg/l}$$

The result may be expressed as:

$$.0038 \text{ mg/l} < P_{true} < .0163 \text{ mg/l*}$$

where $P_{true}$ = the "true" total phosphorus concentration (mg/l).

---

*The confidence limits given here equal ± one standard error. This means that about 68% of the time that the standard error is estimated the "true" value falls within ± one standard error of the predicted value.

Using Table 3.1, Equation 3.21, Equation 3.22a and Table 3A.6, the Reckhow oxic model is applied in a similar manner. The model is (Table 3.1):

$$P = \frac{L}{\frac{18z}{10 + z} + 1.05\ z/\tau\ e^{.012z/\tau}}$$

Substituting:

$$P = \frac{0.12}{\frac{(18)(16.76)}{10 + 16.76} + 1.05(5.238)\ e^{.012(5.238)}}$$

$$= \frac{0.12}{11.273 + 5.857}$$

$$= \frac{0.12}{17.130}$$

$$= .0070 \text{ mg/l}$$

The parameter error contribution may be calculated from the relationship for $s_p^2$ in Table 3A.6. Substituting the values for z and $z/\tau$ for Lake Charlevoix into this relationship, we get:

$$s_p^2 = 7.997 \times 10^{-4}$$

The model error is also available from Table 3.1:

$$s_m = .123,$$

Now, Equation 3.21, with $s'_{iv} = 0$, may be used to estimate the prediction uncertainty:

$$s_{\acute{T}}^2 = \left[P_e - 10^{\log P_e \pm [7.997 \times 10^{-4} + (.123)^2]^{\frac{1}{2}}}\right]^2$$

After substituting the predicted concentration ($P_e$ = .0070 mg/l) into the above relation, we can estimate the confidence limits:*

$$s_{\acute{T}}^2 = \left[.0070 - 10^{\log (.0070) \pm .126}\right]^2$$

$$s_{\acute{T}} = + .0024$$

$$s_{\acute{T}} = - .0018$$

This result may be expressed as:

$$.0052 \text{ mg/l} < P_{true} < .0094 \text{ mg/l}^*$$

For condition 2, the independent variable error, stated earlier, is:

$$s_{\acute{i}v} = .50\ P_e$$

These three error components may now be substituted into Equation 3.21, to estimate the prediction confidence limits* (for the prediction, $P_e$ = .0070 mg/l):

$$s_{\acute{T}}^2 = \left[.0070 - 10^{\log (.0070) \pm [7.997 \times 10^{-4} + (.123)^2]^{\frac{1}{2}}}\right]^2 + [(.50)(.0070)]^2$$

---

*± one standard error.

$$= \left[ .0070 - 10^{-2.155 \pm .126} \right]^2 + 1.225 \times 10^{-5}$$

$$s_{\acute{T}} = + .0042$$

and $s_{\acute{T}} = - .0039$

As in the analysis with the Walker model, this result may be expressed as:

$$.0031 \text{ mg/l} < P_{true} < .0112 \text{ mg/l}^{**}$$

It should be noted that inclusion of the parameter error contribution, $s_p$, had no effect on the confidence limits. This occurs for any lake with independent variable (L, $z/\tau$, and z) values well within the range of values (maximum-minimum) as specified for this model development data set (Table 3A.3).*

Thus, $s_p$ can often be removed from the equation for the prediction confidence limits with little or no effect. Presumably, one will become more adept, with experience, at determining when $s_p$ may be neglected. At a minimum, though, use of the relationship without $s_p$ provides a quick "first-cut" uncertainty estimate. Removing $s_p$, the prediction uncertainty may be estimated from:

$$s_{\acute{T}}^2 = \left[ P_e - 10^{\log P_e \pm s_m} \right]^2 + s_{\acute{i}v}^2 \qquad (3.24a)$$

---

*See the table in the Appendix for a definition of this range for the Reckhow oxic model; the Walker model has a range similar to that for the Reckhow general model.

**± one standard error.

For the Reckhow oxic model, when the independent variable error is proportional to the phosphorus loading, this equation becomes:

$$s_{\acute{T}}^2 = \left[ P_e - 10^{\log P_e \pm .123} \right]^2 + (kP_e)^2 \qquad (3.24b)$$

Equation 3.24b may thus be solved for the confidence limits for the prediction, $P_e$, when an estimate is provided for the loading error proportionality constant, k (if loading error needs to be included).

In summary, Table 3.5, displays the output from the application of the four models. All four models predict the phosphorus concentration relatively well in terms of absolute error (although error varies from 0% to 37%). However, the major point made in Table 3.5 is that only two of the models provide prediction confidence limits. Clearly, the information provided by the application of these two models is more useful in decision making as a result of the uncertainty analysis.

## 3.6 Special Lake Types

For certain types of lakes, the application of the empirical phosphorus lake model is often not too successful. This occurs because of special conditions (e.g., heavy aquatic weed growth) or violation of the assumptions explicit or implicit in the models (e.g., no outflow), or perhaps because a lake is outside or near the bounds on the data sets used to develop the models (e.g., extremely shallow lakes). In this section, some suggestions are made for the treatment of these lake types with a model, while a warning is issued against placing too much emphasis on the model results (until these lake types are more systematically studied).

## TABLE 3.5

## COMPARISON OF THE PREDICTIONS OF THE FOUR MODELS FOR LAKE CHARLEVOIX

Observed Total Phosphorus Concentration = .007 mg/l
(Fall mean and median)

| Model | Predicted Phosphorus Concentration (mg/l) | Prediction Confidence Limits* (mg/l) |
|---|---|---|
| Dillon-Rigler-Kirchner | .0080 | - |
| Larsen-Mercier | .0089 | - |
| Walker | .0096 | $.0064 < P_{true} < .0143$**<br>$.0038 < P_{true} < .0163$*** |
| Reckhow oxic | .0070 | $.0052 < P_{true} < .0094$**<br>$.0031 < P_{true} < .0112$*** |

---

*± one standard error

** no additional loading error

*** additional loading error = ± .5L

Lakes with no measurable outflow (closed lakes) are not uncommon but are not included in the data sets used to construct the models discussed above. Strictly speaking, therefore, these models are not applicable to this lake type. However, it is proposed here that we might "bound" the expected phosphorus concentration with the prediction from two models. First, the lower bound on phosphorus concentration might be estimated by the constant settling velocity model proposed by Chapra (Equation 3.5 with $v_s$ = 16 m/yr):

$$P = \frac{L}{16 + z/\tau} \qquad (3.25)$$

This model is a lower bound since it was developed from a differential equation (3.4) that includes an outlet term, and thus it has a "sink" term that does not exist for closed lakes. $v_s$ = 16 m/yr was selected rather than $v_s$ = 13.2 m/yr (Dillon and Kirchner, 1975) to further extend this lower bound. Note that since $v_s$ = 16 m/yr was estimated for the steady state form in Equation 3.5, for which there is no explicit outlet term, Equation 3.25 may be applicable to closed lakes*

For an upper bound on the phosphorus concentration, consider the differential Equation 3.4 without an outflow term:

$$V\frac{dP}{dt} = M - v_s PA \qquad (3.26)$$

The steady state solution is:

$$P = \frac{L}{v_s} \qquad (3.27a)$$

---

*We cannot state this with any confidence, since closed lakes were not in the model development data set.

Assume here that $v_s$ = 13.2 m/yr (13.2 as opposed to 16 for a conservative upper bound) represents the effective removal of phosphorus-associated material from the lake water column. Then the upper bound on the phosphorus concentration can be estimated from:

$$P = \frac{L}{13.2} \qquad (3.27b)$$

The predicted phosphorus concentration in closed lakes may thus be expressed as:

$$\frac{L}{16 + z/\tau} < P_{true} < \frac{L}{13.2}$$

As an example, consider the data for Lake Charlevoix in Table 3.4 to represent a closed lake. Then:

$$P = \frac{L}{16 + z/\tau} = \frac{0.12}{16 + 5.238} = .0057 \text{ mg/l}$$

and

$$P = \frac{L}{13.2} = \frac{.12}{13.2} = .0091 \text{ mg/l}$$

Thus:

$$.0057 \text{ mg/l} < P_{true} < .0091 \text{ mg/l}$$

Note that this does not include an explicit uncertainty analysis, since we clearly have too little information on closed lakes to quantify model uncertainty. This method proposed for closed lakes must be considered approximate and without firm basis; therefore, the results should be used with caution. It was proposed by the author to fill a gap and prevent application of a black box model alone on a closed lake.

A second lake type that needs brief mention is the extremely shallow lake (z less than perhaps three meters). While the model development data sets for the Reckhow, Walker, and Dillon-Rigler-Kirchner models contain extremely shallow lakes, this investigator has found that this lake type can exhibit unpredictable behavior (in terms of total phosphorus concentration). It is possible that, because of the depth to area ratio and the potential for mixing of the sediments, phosphorus concentration is more variable in shallow lakes (than in deep lakes). On the other hand, these same conditions may prevent the development of anaerobic conditions (which, of course, can increase phosphorus release from the sediments) and reduce concentration variability. Experience in the application of the lake models to this lake type may be most useful here. With experience, the model user may be able to determine factors that affect the success of the models, and the user may then modify the prediction and confidence limits, if needed. In the absence of this modification, the models must be applied cautiously in extremely shallow lakes.

Finally, lakes with extensive aquatic weed growths are probably not modeled well with the techniques proposed here. This should not be surprising since aquatic weed growths in lakes are generally localized, restrict mixing, and interact directly with the sediments. These three conditions suggest violation of the assumptions behind the models. At this point, due to lack of experience with aquatic weed infested lakes, this investigator can offer no suggestions (other than caution) for the use of lake models for this lake type.

## 3.7 Conclusions

A final statement for this chapter on the phosphorus lake models should probably be "apply the models with caution." It has been shown that black box models designed to predict the average annual phosphorus concentration in a lake are relatively successful. Further, uncertainty analysis was introduced as a technique that may be used to estimate confidence limits for the model predictions. However, the models are based on certain assumptions specified in this chapter, and the application of the models is constrained by the limits of the model development data sets. Hence, there is a need for careful consideration in the selection of a model and careful use of the model in lake management.

All the models require an estimate of the phosphorus loading. For many applications, this term will not be available from direct measurement, and thus the literature must be consulted for export coefficients relating land use to annual phosphorus flux. Four sources of this information are reports by Uttormark et al. (1974), Omernik (1976, 1977), and McElroy et al. (1976). Again, careful use is in order. Confidence limits are provided for some of the loading terms, but these apply only for the data gathered and regions studied. Thus, one should "verify" the appropriateness of a particular relationship or export coefficient before application. The ultimate value of a quantitative technique, coefficient, or measurement depends in large part on its uncertainty and applicability. Consideration of that fact will enhance the utility of the material presented in this chapter.

# APPENDIX

## TABLE 3A.1

## MODELS DEVELOPED BY RECKHOW (1977)

| Class | Model | $R^2$ | Standard Error of the Estimate* |
|---|---|---|---|
| Quasi-General | $\frac{L}{11.6 + 1.2\ qs}$ | .915* | .128* |
| Oxic lakes, $z/\tau < 50$ m/yr | $\frac{L}{\frac{18z}{10+z} + 1.05\left(\frac{z}{\tau}\right)e^{.012z/\tau}}$ | .876* | .123* |
| Lakes with $z/\tau > 50$ m/yr | $\frac{L}{2.77z + 1.05\left(\frac{z}{\tau}\right)e^{.0011z/\tau}}$ | .949 | .0088 |
| Anoxic Lakes | $\frac{L}{.17z + 1.13z/\tau}$ | .948* | .105* |

*Based on logarithmic transformation of terms.

## TABLE 3A.2

## STATISTICS FOR 95 NORTH TEMPERATE LAKES (RECKHOW, 1977)

| Variable | Minimum | Geometric Mean | Maximum |
|---|---|---|---|
| z (m) | 0.78 | 7.8 | 100.0 |
| $\tau$ (yr) | 0.003 | 0.47 | 57.6 |
| $z/\tau$ (m/yr) | 0.53 | 16.5 | 575.0 |
| $L\tau/z$ (mg/l) | .0075 | .066 | 1.179 |
| P (mg/l) | .004 | .033 | .958 |
| $R_p$ | -.08 | .38* | .95 |

Correlation Coefficients

| Variable | log $\tau$ | log $(z/\tau)$ | log $(L\tau/z)$ | log P | $R_p$ |
|---|---|---|---|---|---|
| log z | .751 | -.379 | -.151 | -.408 | .508 |
| log $\tau$ | | -.896 | .119 | -.249 | .700 |
| log $z/\tau$ | | | -.260 | .074 | -.639 |
| log $L\tau/z$ | | | | .851 | .049 |
| log P | | | | | -.405 |
| $R_p$ | | | | | |

*arithmetic mean

## TABLE 3A.3

## STATISTICS FOR THE THREE MODEL CLASSES

| Variable | Minimum | Geometric Mean | Maximum |
|---|---|---|---|
| | Oxic Lakes, $z/\tau < 50$ m/yr (33 lakes) | | |
| $z$ (m) | 0.73 | 18.2 | 100.0 |
| $\tau$ (yr) | 0.065 | 2.57 | 57.6 |
| $z/\tau$ (m/yr) | 0.96 | 7.1 | 48.6 |
| $L\tau/z$ (mg/l) | .008 | .040 | .298 |
| P (mg/l) | .004 | .013 | .060 |
| $R_p$ | .11 | .56* | .95 |
| | Lakes with $z/\tau > 50$ m/yr (28 lakes) | | |
| $z$ (m) | 0.85 | 4.05 | 13.4 |
| $\tau$ (yr) | 0.003 | 0.028 | 0.252 |
| $z/\tau$ (m/yr) | 51.8 | 147.0 | 575.0 |
| $L\tau/z$ (mg/l) | .008 | .047 | .178 |
| P (mg/l) | .007 | .034 | .135 |
| $R_p$ | -.08 | .18* | .37 |
| | Anoxic Lakes (21 lakes) | | |
| $z$ (m) | 1.40 | 6.9 | 34.0 |
| $\tau$ (yr) | 0.008 | 0.52 | 4.5 |
| $z/\tau$ (m/yr) | 3.3 | 13.3 | 425.0 |
| $L\tau/z$ (mg/l) | .024 | .119 | .621 |
| P (mg/l) | .017 | .088 | .610 |
| $R_p$ | -.03 | .23* | .56 |

*arithmetic mean

## TABLE 3A:4

## PARAMETER STATISTICS FOR THE RECKHOW MODELS

| Parameter | Standard Deviation | Correlations | | | |
|---|---|---|---|---|---|
| Oxic, $z/\tau < 50$ m/yr lakes | | 18.0 | 10.0 | 1.05 | 0.12 |
| 18.0 | $2.41 \times 10^{-3}$ | | .538 | -.274 | .133 |
| 10.0 | $6.56 \times 10^{-3}$ | | | .028 | .735 |
| 1.05 | $9.27 \times 10^{-2}$ | | | | -.351 |
| 0.12 | $5.45 \times 10^{-3}$ | | | | |
| Lakes with $z/\tau > 50$ m/yr | | | 2.77 | 1.05 | .0011 |
| 2.77 | 1.24 | | | -.650 | .415 |
| 1.05 | $9.27 \times 10^{-2}$ | | | | .889 |
| .0011 | $4.37 \times 10^{-4}$ | | | | |
| Anoxic Lakes | | | | 1.13 | |
| 0.17 | $2.61 \times 10^{-2}$ | | | -.656 | |
| 1.13 | $3.95 \times 10^{-2}$ | | | | |

## TABLE 3A.5

## PARAMETER STATISTICS FOR THE WALKER MODEL

| Parameter | Standard Deviation | Correlation Coefficient .546 |
|---|---|---|
| .824 | .067 | -0.0064 |
| .546 (= 1 - .454) | .046 | |

Prediction Error Contribution from Parameter Uncertainty

$$s_p^2 = 0.001\ Y^4\ \tau^{.908}\ [4.49 + 1.44\ (\ln\ \tau)^2 + 0.032\ \ln\ \tau)]$$

where $Y = 1/(1 + .824\tau^{.454})$

ln = base e logarithm

## TABLE 3A.6

## PREDICTION ERROR CONTRIBUTION FROM PARAMETER UNCERTAINTY FOR THE MODEL PROPOSED BY RECKHOW

## FOR OXIC LAKES WITH $z/\tau < 50$ m/yr

$$s_p^2 = [(\frac{1}{B})\ (z/\tau)(e^{.012z/\tau})(.0927)]^2 + [(\frac{1}{B})(1.05(z/\tau)^2 e^{.012z/\tau})(.00545)]^2$$

$$- 2(\frac{1}{B})^2(z/\tau)^3(e^{.012z/\tau})^2(1.05)(.351)(.0927)(.00545)$$

where

$$B = \frac{18z}{10 + z} + 1.05\ (z/\tau)e^{.012z/\tau}$$

## Uncertainty Analysis

The technique used to estimate the prediction confidence limits is derived from the Taylor series expansion, which provides an approximation for a function, f(x). For a bivariate relationship, the Taylor series may be written as:

$$f(x) = f(x_o) + \frac{\partial f(x)}{\partial x}(x - x_o) + \frac{1}{2}\frac{\partial^2 f(x)}{\partial x^2}(x - x_o)^2 + \ldots \qquad (3A.1)$$

If it may be assumed that the error $(x - x_o)$ is small and the higher derivatives of f(x) are not large (the relationship is not highly nonlinear in the region of interest), then Equation 3A.1 reduces to:

$$f(x) - f(x_o) \simeq \frac{\partial f(x)}{\partial x}(x - x_o) \qquad (3A.2)$$

For a multivariate relationship, this becomes:

$$f(x_i,x_j,x_k) - f(x_{io},x_{jo},x_{ko}) \simeq \frac{\partial f(x_i,x_j,x_k)}{\partial x_i}(x_i - x_{io}) + \frac{\partial f(x_i,x_j,x_k)}{\partial x_j}(x_j - x_{jo}) + \frac{\partial f(x_i,x_j,x_k)}{\partial x_k}(x_k - x_{ko}) \qquad (3A.3)$$

Squared and expressed in standard error form, Equation 3A.3 may be converted to the error propagation equation given below.

$$s(P) \simeq \sum_{i=1}^{n}\left[\left(\frac{\partial P}{\partial x_i}\right)^2 s^2(x_i) + \sum_{j=i+1}^{n} 2\,\frac{\partial P}{\partial x_i}\,\frac{\partial P}{\partial x_j}\,s(x_i)s(x_j)\rho(x_i,x_j)\right]^{1/2} \qquad (3A.4)$$

$s(x_i)$ is the estimate of the standard deviation of $x_i$, and $\rho(x_i,x_j)$ is the correlation between $x_i$ and $x_j$. Berthouex (1975) and Cornell (1972) discuss this error analysis approach for sanitary engineering and hydrologic applications, respectively.

Equation 3A.4 may be used to estimate the effect of model parameter error and input variable error. This was done for the Reckhow oxic lakes ($z/\tau < 50$ m/yr) model, and the result is presented in Table 3A.6. These error terms are then combined with the standard error of the model for an estimate of the prediction confidence limits. The empirical models presented in Chpater 3 are linear (or nearly linear) in L and approximately linear in $z/\tau$ for most lakes. Since the variance in z is usually quite small, it is ignored. Therefore, Equation 3A.4 may be used with an empirical model (providing model and parameter error estimates) for all lakes within the model's constraints, except for those lakes with a highly non-linear relation between P and $z/\tau$ (over the estimated variance in $z/\tau$ for that lake). This nonlinearity should be checked before application of Equation 3A.4; otherwise, Equation 3A.4 may significantly overestimate the total error.

The model error,and the contributions to the total prediction uncertainty from the parameter error and the independent variable uncertainty,are normally combined in variance form:

$$s_T^2 = s_m^2 + s_p^2 + s_{iv}^2 \qquad (3A.5)$$

where

$s_T$ = total error (logarithmic form)

$s_p$ = parameter error contribution to the total error (log form)

$s_m$ = standard error of the estimate for the model (log form)

$s_{iv}$ = input variable error (log form) contribution to total error

$s_m$ for the models discussed herein are presented in Tables 3A.1 and 3.3, and $s_p$ may be calculated for each lake from Equation 3A.4 and the statistics presented in Tables 3A.4 and 3A.5. $s_{iv}$ results from uncertainty in the phosphorus loading and in the hydraulic detention time. $s_{iv}$ would, therefore, be estimated using the error propagation equation for the model *variables*, just as the error propagation equation was used for the model *parameters* to derive the relationship for $s_p$. However, for most lakes, the major component of the variable error results from phosphorus loading, and often the loading uncertainty is estimated as *proportional* to the loading. That approach was adopted for the applications presented in this chapter. Since the models discussed herein are linear in L, the loading uncertainty is proportional to the phosphorus concentration. Since the loading uncertainty is proportional to the concentration rather than to the log of the concentration, it is easier, for calculation purposes, to include this uncertainty term in the form:

$$s'_{iv} = kP$$

where

$s'_{iv}$ = untransformed uncertainty component due to the phosphorus loading

k = a constant

The model applied here was developed from a logarithmic transformation of terms. For this reason, Equation 3A.5 must refer to the log-transformed model. The error terms are thus combined in the following way:

$$10^{s_T^2} = s_{T}'^{2} = \left[ P - 10^{\log P \pm [(s_p)^2 + (s_m)^2 + (s_{iv})^2]^{1/2}} \right]^2 \qquad (3A.7)$$

For application purposes, this is rewritten:

$$s_{T}'^{2} = \left[ P - 10^{\log P \pm (s_p^2 + s_m^2)^{1/2}} \right]^2 + s_{iv}'^{2} \qquad (3A.8)$$

# CHAPTER 4

# LAKE CLASSIFICATION

## 4.1 Introduction

The models presented in Chapter 3 may be used to predict the phosphorus concentration in a lake. An alternative way to model or quantify the effect of phosphorus (nutrient) loading or concentration on lake quality is through lake classification criteria. Much of the same information is conveyed through the application of either the models or the classification criteria, but the information is presented in different formats. The models may be used to estimate the concentration of a single constituent (phosphorus) on a continuous scale. Classification criteria, on the other hand, may be used to group lakes into a small number of discrete quality (trophic) states. This results in an easily interpretable "lake-labeling" scheme which has the advantages and disadvantages of any classification system designed to reduce the dimensionality or discretize a set of continuous data.

Lake classification schemes actually include two distinct approaches. The first, often identified with trophic state criteria, classify a lake according to the current quality of the water column (the water in the lake) using data from in-lake samples. These typically require measurements of total phosphorus concentration, total nitrogen concentration, chlorophyll a level, Secchi disk depth and/or other constituents as well (all taken within the lake). The second approach, generally identified with nutrient (phosphorus) loading criteria, can be used to predict the quality state of a lake from data on the phosphorus loading and lake geomorphology (no in-lake measurements). Loading criteria can be used to forecast the effect of changes in phosphorus loading on lake quality. Further, they can be reformulated and used to estimate the phosphorus carrying capacity associated with each quality state. In summary, the trophic state criteria can provide a multivariable index of present water quality, while the loading criteria focus on a single nutrient but can be used to predict quality changes over time and estimate carrying capacity.

This chapter presents discussions of both approaches. The emphasis is placed on loading criteria, however, because of the usefulness of that information for planning and management of lake quality. Case studies are presented illustrating some applications.

## 4.2 Trophic State Criteria

A number of attempts have been made to establish a trophic state criterion as a function of commonly measured water quality variables. The EPA National Eutrophication Survey (1974a) has compared the work of some investigators (Sakamoto, 1966; National Academy of Sciences, 1972; and Dobson et al., 1974) on chlorophyll a levels versus trophic state. This is presented in Table 4.1. The EPA's own estimates of values of chlorophyll a, total phosphorus, and Secchi disk depth indicative of trophic states are presented in Table 4.2.

While there have been other attempts at single variable trophic state criteria (or indices), all are relatively similar in approach (see Table 4.1). More importantly, they represent subjective judgment,

## TABLE 4.1

## TROPHIC STATE VS. CHLOROPHYLL a

(from EPA-NES 1974a)

| Trophic Condition | Chlorophyll a (μg/l) Sakamoto | Academy | Dobson | EPA-NES |
|---|---|---|---|---|
| Oligotrophic | 0.3-2.5 | 0-4 | 0-4.3 | <7 |
| Mesotrophic | 1-15 | 4-10 | 4.3-8.8 | 7-12 |
| Eutrophic | 5-140 | >10 | >8.8 | >12 |

## TABLE 4.2

## EPA-NES TROPHIC STATE DELINEATION

(from EPA-NES 1974a)

| Trophic State | Chlorophyll a (μg/l) | Total Phosphorus (μg/l) | Secchi Disc Depth (m) |
|---|---|---|---|
| Oligotrophic | <7 | <10 | >3.7 |
| Mesotrophic | 7-12 | 10-20 | 2.0-3.7 |
| Eutrophic | >12 | >20 | <2.0 |

and possibly limited geographic regions, so it is unlikely that universal agreement will rest on one approach. Therefore, the selection of a univariate trophic state criterion should be based primarily on personal acceptance and credibility.

More robust trophic state criteria or indices may be developed with a multivariate approach. Shannon and Brezonik (1972) developed a trophic index for Florida lakes composed of the variables: primary production (PP, in mg. of carbon per cubic meter-hour), chlorophyll a (CHA, in $mg/m^3$), total organic nitrogen (TON, in mg/l as N), total phosphorus (TP, in mg/l as P), Secchi disk transparency (SD, in meters), specific conductance (COND, in $\mu$mho/cm), and a cation ratio (CR, a dimensionless ratio of (Na + K)/(Ca + Mg)). For lakes without appreciable organic color, the trophic state index (TSI) was estimated as:

$$\begin{aligned} TSI = {} & 0.936\,(1/SD) + 0.827\,(COND) + 0.907\,(TON) \\ & + 0.748\,(TP) + 0.938\,(PP) + 0.892\,(CHA) \\ & + 0.579\,(1/CR) + 4.76 \end{aligned} \tag{4.1}$$

A TSI of about 3 to 5 defines the transition zone between eutrophy and mesotrophy, and a TSI of 1.2 to 1.3 separates the mesotrophic and oligotrophic classes.

The index was developed using principal component analysis, and the TSI is the first principal component. This technique may be used to identify "common elements" among variables, and the first principal component is a linear combination of the variables that best describes the most common element. When all of the variables in an analysis are thought to be good indicators of a concept called trophic state, then it is reasonable to assume that the most common element extracted from this set of variables (the first principal component) would be a good index of trophic state. In fact, this component is more "robust" than any one variable as an indicator of trophic state. This means that it is less likely, than a single variable index, to misclassify a lake based on an erroneous measurement. Incorrect data on one variable can lead to misclassification based on that variable, but it may not lead to misclassification if the classification criterion is based on other variables (correctly measured) as well.

Despite the fact that a principal component trophic state index has this desirable feature of robustness, the TSI proposed by Shannon and Brezonik cannot be recommended for use on north temperate lakes. The TSI was developed from a data base of Florida lakes only, and the significant climatic (and thus thermal) difference between that area and the north temperate region is likely to affect the index. Since this effect is unclear, we are unable to interpret the TSI in north temperate lakes. Equally important, most of the trophic variables are log-normally distributed (see Chapter 2 for an explanation), which means that the best estimate for the TSI should be made under a logarithmic transformation for these variables. Without this transformation (as in the case of Shannon's and Brezonik's TSI), the index may be less precise. Therefore, while principal component analysis is a useful technique for defining a trophic state index, the TSI developed by Shannon and Brezonik is inappropriate for north temperate lakes.

A trophic state index has recently been proposed by Carlson (1977) that may also be considered multivariate. Carlson's index may be estimated from summer values of Secchi disk depth, summer total phosphorus concentration, or summer chlorophyll a concentration (or a weighted combination of all three). Carlson used regression analysis to relate Secchi disk depth to total phosphorus concentration and to chlorophyll a concentration. He then reasoned that a doubling of biomass levels, or a halving of the Secchi disk depth, corresponds to a change in trophic state. Carlson arbitrarily assigned a TSI scale of 0-100 to the three trophic variables, such that a change of 10 units in TSI corresponds to a halving of the Secchi disk depth and a change in trophic state. The regression equations then were used to relate the TSI to phosphorus and chlorophyll. Table 4.3 contains the index values and variable relationships.

Carlson's TSI may be estimated from any of the three variables, using Table 4.3. Carlson felt that this was important as:

1. Secchi disk readings may be misleading as a trophic state indicator in colored lakes or highly turbid (non-algal) lakes.

2. Chlorophyll a may be the best indicator during the growing season.

3. Phosphorus may not be a good indicator in non-phosphorus limited lakes.

Thus different variables may be used depending upon the season, lake, and availability and quality of data. While Carlson suggests that

## TABLE 4.3

## CARLSON'S TROPHIC STATE INDEX

| TSI | Secchi disk (m) | Surface phosphorus (mg/m$^3$) | Surface chlorophyll (mg/m$^3$) |
|---|---|---|---|
| 0 | 64 | 0.75 | 0.04 |
| 10 | 32 | 1.5 | 0.12 |
| 20 | 16 | 3 | 0.34 |
| 30 | 8 | 6 | 0.94 |
| 40 | 4 | 12 | 2.6 |
| 50 | 2 | 24 | 6.4 |
| 60 | 1 | 48 | 20 |
| 70 | 0.5 | 96 | 56 |
| 80 | 0.25 | 192 | 154 |
| 90 | 0.12 | 384 | 427 |
| 100 | 0.062 | 768 | 1,183 |

the variable that the index is based on be selected on a pragmatic basis, he recommends that consideration be given to chlorophyll in the summer and to phosphorus in the fall, winter, and spring.

A comparison of Carlson's TSI and the indices proposed in Tables 4.1 and 4.2 (under the assumption that a lake with Carlson's TSI $< 40$ is oligotrophic and a lake with Carlson's TSI $> 50$ is eutrophic) shows a great deal of similarity, which should be expected since the relationships among Secchi disk depth, chlorophyll levels, and phosphorus levels are well established and understood. Thus Carlson's major contribution is not new insight on trophic state divisions, nor a multivariate index*, but rather the expression of the index on a 0-100 scale. This effectively eliminates the subjective labeling of oligotrophic, mesotrophic, and eutrophic states, and substitutes a scale easily interpretable by the layman. This advantage may be removed, however, by the tendency of researchers to mentally convert Carlson's TSI value back to one of the three standard trophic states for ease of interpretation. In summary, then, the selection of a trophic state index from among those given in Tables 4.1 - 4.3 should probably be made on the basis of familiarity by the users, since no single index conveys appreciably more information than any of the others.

## 4.3 Phosphorus Loading Criteria

### 4.3.1 Early Development

The development of nutrient loading criteria, in particular phosphorus loading criteria for lakes, centers around Vollenweider's work. Vollenweider's initial work on this topic (1968) was derived from the well-documented inverse relationship between mean depth and various measures of quality or biomass. Based on available data and his subjective judgment on lake quality, Vollenweider estimated "permissible" and "dangerous" (2x permissible) annual areal nutrient loading levels as a function of mean depth. These loading criteria are contained in Table 4.4.

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*The EPA's index in Table 4.2 can be considered multivariate, since the trophic state may be based on any or all of the three variables.

However, it became apparent that hydraulic flushing rates also influence the capacity of a lake to accept nutrient loading. Lack of consideration of this factor may result in an underestimation of the trophic state for lakes with high flushing rates (low detention times). Therefore, Vollenweider (1973) modified his original work, and he selected overflow rate ($z/\tau$) as the best determinant (along with areal nutrient loading) of lake quality across a wide range of lake classes. This time, he again subjectively established "permissible" and "dangerous" loading levels (as a function of $z/\tau$ and L). Despite this subjectivity, these loading criteria are attractive because they may be easily applied to assess the susceptibility of a lake to phosphorus input. As a result, they have been used extensively. A plot of this second loading criterion is given in Figure 4.1.

At about the same time Vollenweider developed his revised loading criterion, Dillon (1975) also recognized the importance of detention time or flushing rate. As a result of analyses performed on two Canadian Shield Lakes, Dillon proposed a phosphorus loading criterion graph with $L\tau(1 - R_p)$ on one axis and z on the other axis. Dillon's phosphorus loading criterion is given in Figure 4.2. This criterion

## TABLE 4.4

## PERMISSIBLE LOADING LEVELS FOR TOTAL NITROGEN AND TOTAL PHOSPHORUS (BIOCHEMICALLY ACTIVE)*

($g/m^2$-yr)

| Mean Depth up to | Permissible Loading, up to | | Dangerous Loading, in excess of | |
|---|---|---|---|---|
| | N | P | N | P |
| 5 m | 1.0 | 0.07 | 2.0 | 0.13 |
| 10 m | 1.5 | 0.10 | 3.0 | 0.20 |
| 50 m | 4.0 | 0.25 | 8.0 | 0.50 |
| 100 m | 6.0 | 0.40 | 12.0 | 0.80 |
| 150 m | 7.5 | 0.50 | 15.0 | 1.00 |
| 200 m | 9.0 | 0.60 | 18.0 | 1.20 |

*After Vollenweider (1968).

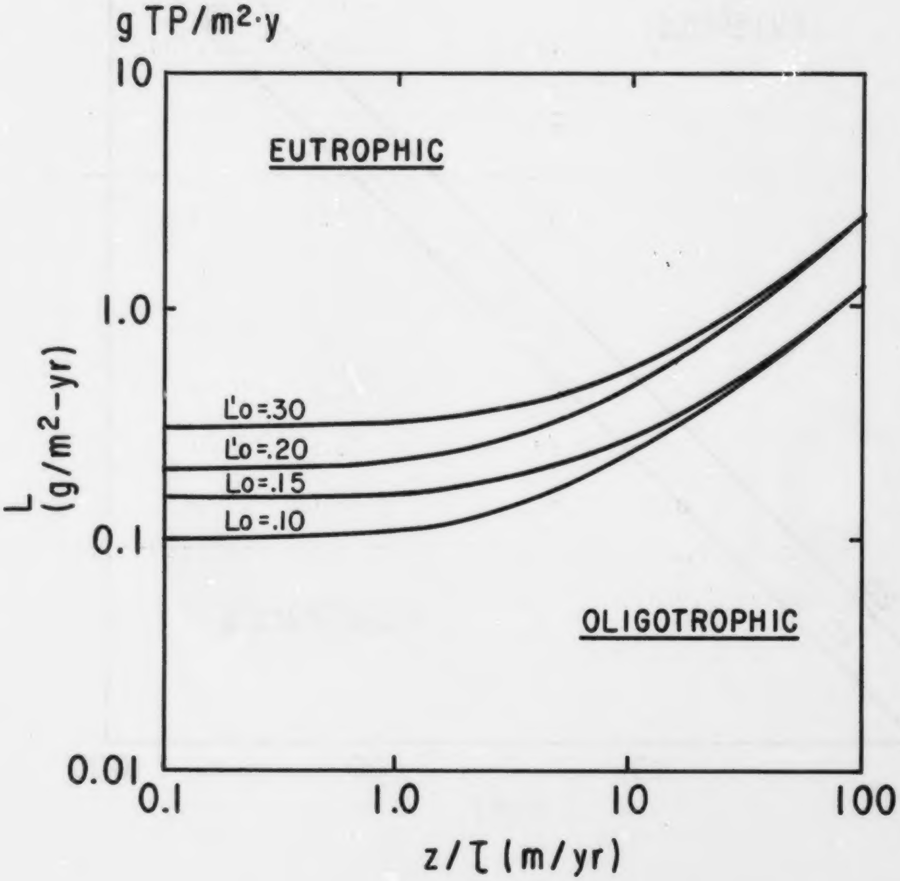

Figure 4.1 Vollenweider's Phosphorus Loading Criterion (1975), L versus $q_s$ $(z/\tau)$

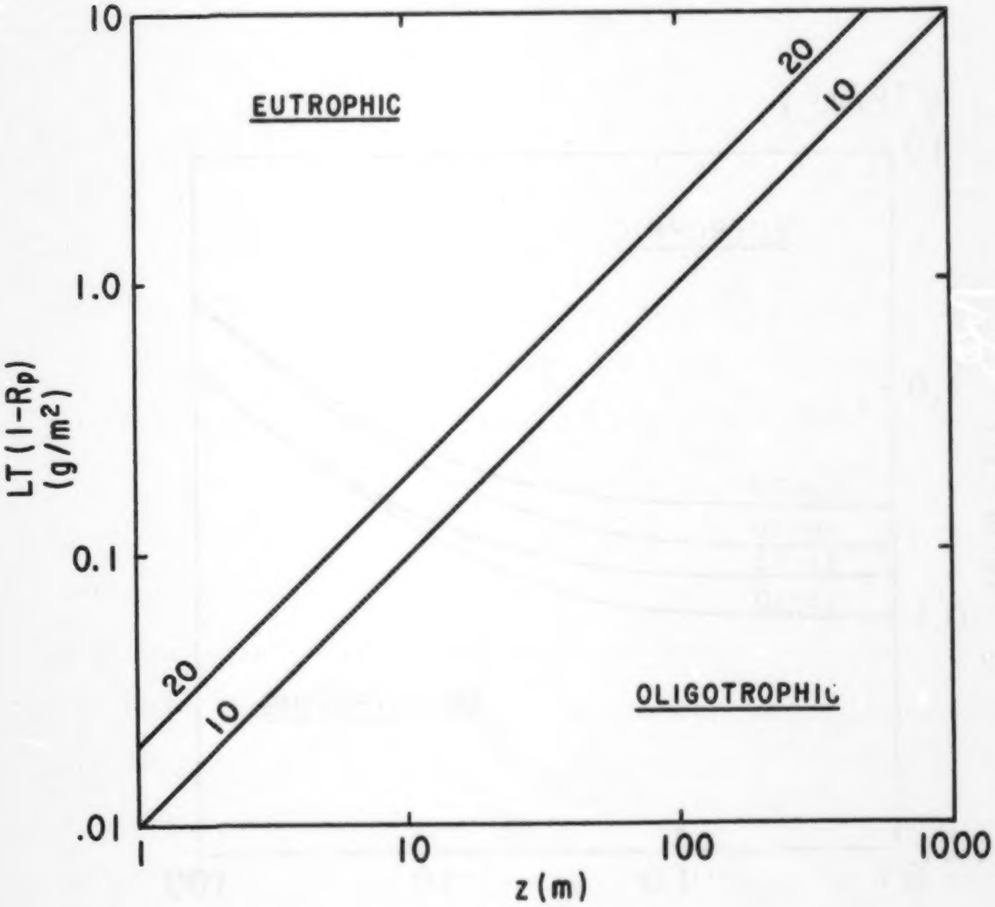

Figure 4.2 Dillon's Phosphorus Loading Criterion, With Lines of Constant Phosphorus Concentration (in μg/l) Distinguishing Trophic States (From Dillon, 1975)

accounted for the effect of detention time, and it resulted in trophic state delineation by lines of constant phosphorus concentration. This becomes apparent when it is noted that the definition of the outlet phosphorus concentration is:

$$P = \frac{L\tau}{z}(1 - R_p) \qquad (4.2)$$

Thus Dillon's criterion is actually a plot of this definition for the outflow concentration. However, application of this criterion is possible only with an estimate for $R_p$ [available later using the empirical model of Kirchner and Dillon (1975)] independent of phosphorus budget considerations; otherwise application of Dillon's criterion requires as input the variable it is designed to estimate (or classify), the total phosphorus concentration.

Recently, Larsen and Mercier (1975) in conjunction with the EPA-NES (EPA National Eutrophication Survey), proposed another variation of the phosphorus loading criterion graph. They plotted mean influent phosphorus concentration versus the measured phosphorus retention coefficient, and distinguished trophic states using lines of constant phosphorus concentration. Their approach is presented in Figure 4.3. This criterion plot illustrates the interplay between the lake basin impact (influent phosphorus concentration) on the lake quality, and the lake's capacity to absorb that impact (the phosphorus retention coefficient). Like Dillon's plot, the Larsen-Mercier graph is a direct result of Equation 4.2, and thus is subject to the same comments. (Larsen and Mercier have also proposed an empirical relation for $R_p$.) However, Larsen and Mercier felt that their graph was more

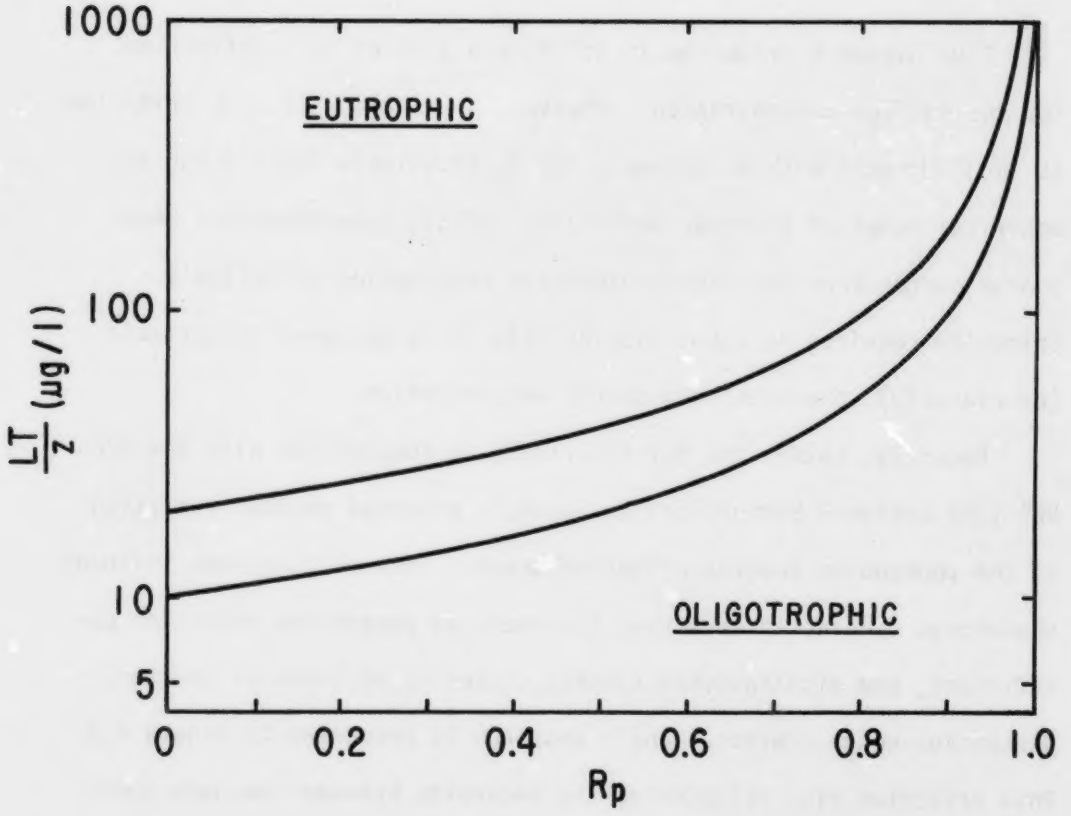

Figure 4.3 The Larsen-Mercier Phosphorus Loading Criterion (From Larsen and Mercier, 1975)

informative, since it separates the effects of the two major determinants of lake phosphorus concentration (influent concentration and retention coefficient).

In summary, with the exception of Vollenweider's first loading criterion (z vs. L) which did not explicitly consider variations in lake flushing rate (and thus may be misleading, particularly for lakes with high flushing rates), the phosphorus loading criteria discussed in this section predict trophic state about equally well for a set of lakes. The choice of which of these three to use, may, therefore, be a function of preferred form or familiarity. However, before selecting one of these three loading criteria, Sections 4.3.2, 4.3.4, and 4.3.5 should be read for more recent developments. Specifically, consideration of an additional variable, consideration of uncertainty, and development of "optimal," objective loading criteria are discussed. While not in common use yet, these new phosphorus loading criteria represent improvements in the development and application of this technique, and thus they should be given serious consideration.

### 4.3.2 Recent Work By Vollenweider

Vollenweider (1976) recently proposed new phosphorus loading criteria that build upon his previous work and the work of others. Most of the loading criteria discussed in the last section are used to examine the relation between phosphorus loading and trophic state as a function of mean depth (z), hydraulic detention time ($\tau$), or surface overflow rate ($q_s$), individually. In this recent paper, Vollenweider proposes that, in general, refinements in trophic state assessment can be made if any two of these variables ($z$, $\tau$, and $q_s$) are considered together.

Vollenweider empirically derived a model to estimate "critical" loading (the transitional loading between the oligotrophic and mesotrophic states):

$$L_c = P_c\, z \left(\frac{1 + \sqrt{\tau}}{\tau}\right) \tag{4.3}$$

where

$L_c$ = critical phosphorus loading (g/m$^2$-yr)
$P_c$ = critical phosphorus concentration (mg/l)
$z$ = lake mean depth (m)
$\tau$ = hydraulic detention time (yr)

This is equivalent to:

$$L_c = P_c\, q_s\, (1 + \sqrt{z/q_s}) \tag{4.4}$$

where

$q_s = z/\tau$ = surface overflow rate (m/yr)

Further, if Equation 4.3 is rearranged so that phosphorus concentration is on the left side of the equation, then:

$$P = \frac{L\tau}{z}\left(\frac{1}{1 + \sqrt{\tau}}\right) = P_{in}\left(\frac{1}{1 + \sqrt{\tau}}\right) \tag{4.5}$$

where

$\frac{L\tau}{z} = P_{in}$ = average influent phosphorus concentration (mg/l)

With additional rearranging of terms in Equation 4.5, we can derive the Larsen-Mercier phosphorus model from Chapter 3:

$$P = \frac{L\tau}{z}\left(1 - \frac{1}{1 + 1/\sqrt{\tau}}\right) \qquad (4.6)$$

Vollenweider used Equations 4.3 and 4.4 to construct his new graphical loading criterion. It is interesting to note that these equations are equivalent in form to the Larsen-Mercier model, which was used to construct the Larsen-Mercier phosphorus loading criterion.*

Vollenweider's new phosphorus loading criterion, developed using either Equation 4.3 or 4.4 with a critical phosphorus concentration of .010 mg/l, is presented in Figure 4.4. Note that it is plotted on the same axes, L vs. $q_s$, used for Vollenweider's second criterion (Figure 4.2). The difference between these two criteria is that Figure 4.4 may be used to estimate the additional effect of mean depth or hydraulic detention time (trhough a series of lines representing the critical loading). Vollenweider provides the following example for the application of Figure 4.4. For Lake Ontario, with $z = 84$ m, $\tau = 7.9$ yr, and $q_s = 10.6$ m/yr, start from $q_s = 10.6$ on the horizontal axis. Proceed vertically up the imaginary $q_s = 10.6$ line to a point, within the grid of z and $\tau$ lines, that best represents $z = 84$ m and $\tau = 7.9$ yr. Then, from that point proceed horizontally to the vertical axis and read the critical loading value. For Lake Ontario, critical loading (the transition between oligotrophic and mesotrophic) is 350-400

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*Thus, Vollenweider, unlike Larsen and Mercier, is able to examine the effect of both $q_s$ and z because he "removed" qs from $P_{in}$ and expressed loading as an areal term (L). However, it is unclear which approach is more informative.

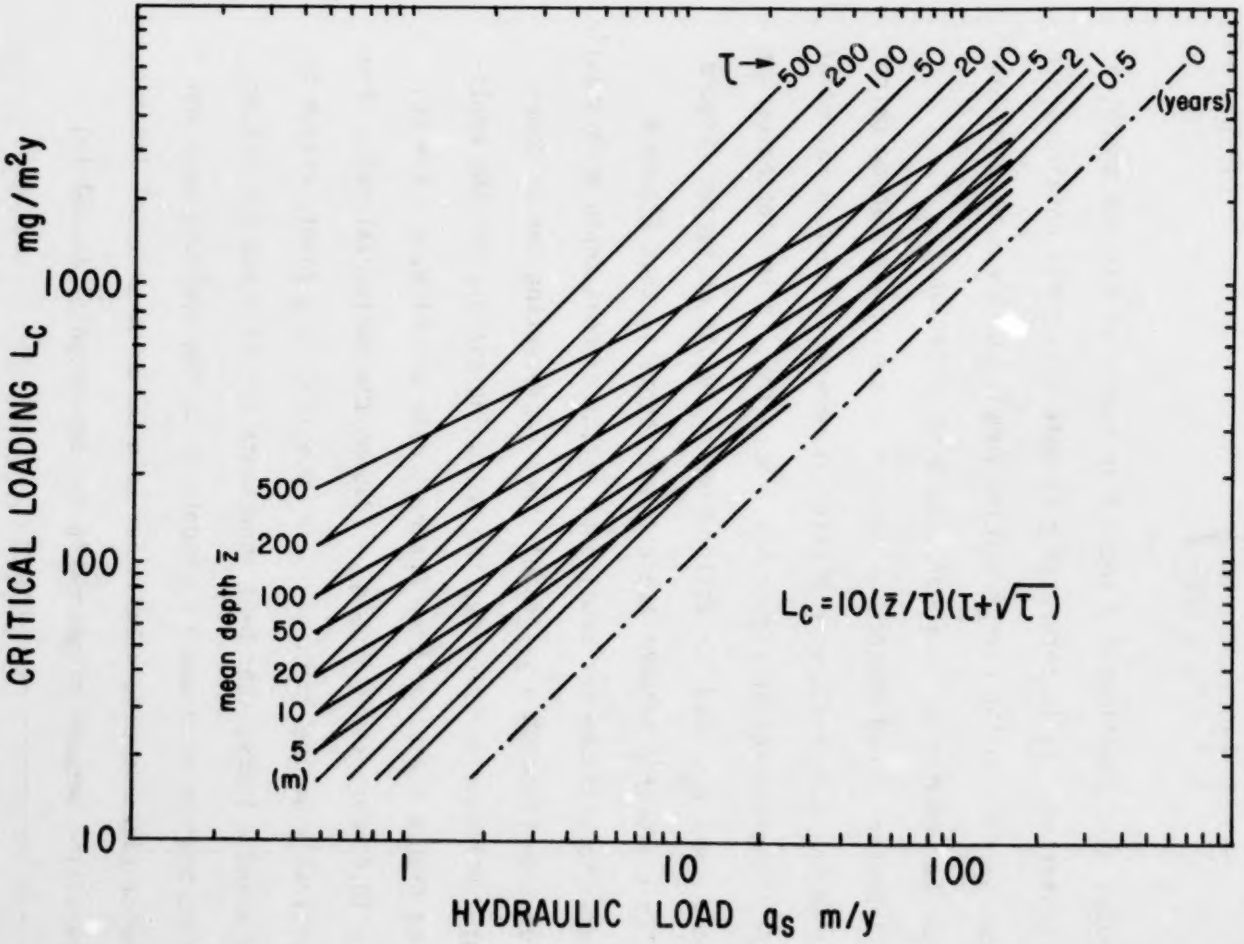

Figure 4.4 Vollenweider's Phosphorus Loading Criterion (1976), L versus $q_s$, z, and $\tau$

mg/m$^2$-yr. Eutrophic loading, according to Vollenweider, is then simply double the critical loading.

This approach represents a restatement of the loading criteria discussed in the last section. By graphically including $q_s$ and either $z$ or $\tau$ in the trophic state assessment, an easy-to-use procedure was developed that appears to be sensitive to changes in both $q_s$ and $z$, or both $q_s$ and $\tau$ (which, if the procedure is valid, should make the analysis more robust). Future applications of this criterion will serve to check its usefulness and reliability in trophic state assessment.

### 4.3.3 A Lake Classification Scheme Proposed by Uttormark and Wall

Uttormark and Wall (1975) proposed a subjective "Lake Condition Index," LCI, that actually is a trophic state index, but it is included in this section because of the applications suggested by Uttormark and Wall, when the LCI is coupled with phosphorus loading information. Briefly, the LCI is constructed from easily detected measures of eutrophication. Penalty points are assigned according to Table 4.5. The sum of the penalty points is the LCI; higher values indicate poorer quality. Uttormark and Wall provide more detailed tables in their report for specifying the points more precisely.

Uttormark and Wall plotted lakes on a graph of LCI (which is used to estimate present lake quality) versus areal phosphorus loading (in g/m$^2$-yr, and which may be considered an estimate of potential or future quality). They reasoned that the position of a lake on this plot, shown in Figure 4.5, would indicate the type of management strategy that would be most effective for that lake. Figure 4.5 may be divided

## TABLE 4.5

## POINT SYSTEM FOR LAKE CONDITION INDEX*

| Parameter | Points |
|---|---|
| Hypolimnetic dissolved oxygen | 0 - 6 |
| Transparency | 0 - 4 |
| Fishkills | 0 - 4 |
| Use impairment (extent of macrophyte or algal growths) | 0 - 9 |
| Total | 0 - 23 |

*From Uttormark and Wall (1975)

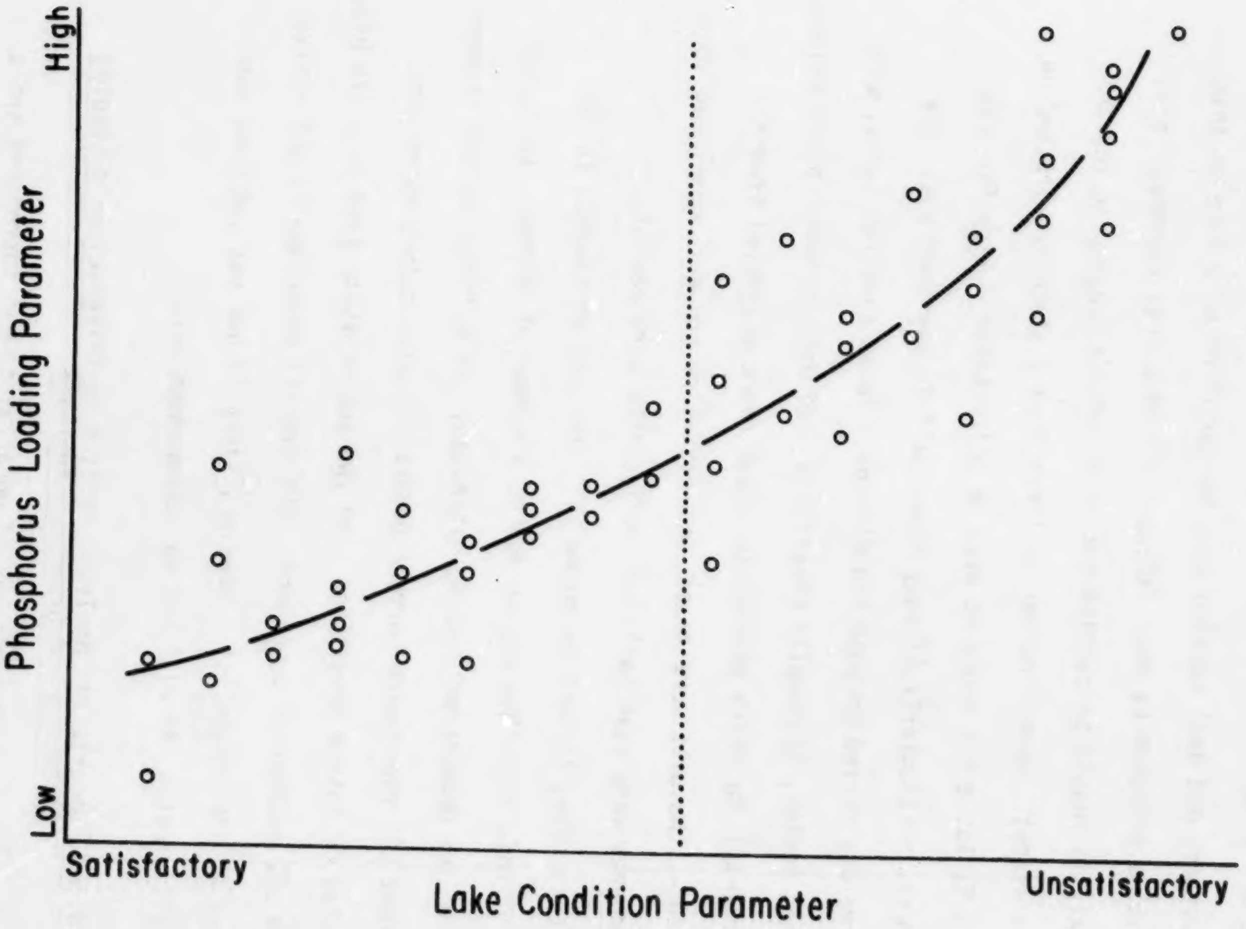

Figure 4.5 Uttormark's and Wall's Lake Classification Scheme (From Uttormark and Wall, 1975)

into four quadrants, and the management strategies appropriate to each quadrant are given in Table 4.6.

Uttormark and Wall caution that the position of a lake on this curve can be affected by many factors, and thus they recommend that this analysis should be carried out on a limited geographic region (e.g., a state). When a number of lakes from a state are plotted on Figure 4.5, Table 4.6 might be used to guide state policy for lake management, particularly if many lakes fall in one quadrant. The procedure was tested on selected Wisconsin lakes (natural lakes, with inlet and outlet, seasonally stratified, and with no known point sources of nutrients) for which phosphorus inputs were estimated from the literature. Results were mixed, and Uttormark and Wall concluded that the concepts were reasonable but refinements were needed.

In summary, it must be noted that the lack of success in the trial of this technique may be due to a number of factors, including error in the phosphorus loading estimates, the validity of the assumption that LCI represents current conditions and loading represents potential or future conditions, and the subjectivity (and possible bias) of the LCI estimation procedure. The overall technique is attractive because of its simplicity. However, since it has not yet been successfully tested, it will not be recommended here.

### 4.3.4 Uncertainty Analysis Applied to Vollenweider's Loading Criterion - A Graphical Approach

In Chapter 3, the concept of uncertainty was introduced and a procedure was developed for the estimation of confidence limits on a model prediction. The same idea is equally important in this chap-

## TABLE 4.6

## SUGGESTED MANAGEMENT STRATEGIES ASSOCIATED WITH THE FOUR QUADRANTS IN FIGURE 4.5*

| Set | Phosphorus Loading | Lake Condition | General Water Quality Management Grouping |
|---|---|---|---|
| a | Low | Satisfactory | No present danger |
| b | Low | Unsatisfactory | Renewal desirable; long-term benefits may be possible without extensive nutrient abatement |
| c | High | Satisfactory | Prompt protection needed; degradation may be imminent |
| d | High | Unsatisfactory | Problem lakes; renovation desirable but lasting improvement may require extensive nutrient abatement |

*From Uttormark and Wall (1975)

ter. That is, there is uncertainty in the estimates of L, z, $\tau$, and $q_s$. There is also uncertainty in the subjective estimates of trophic state transition lines (loading criteria). As in Chapter 3, there is clearly a need here to estimate this uncertainty,and to use this total uncertainty estimate as a weight associated with a prediction or trophic state assessment.

A graphical technique has been developed (Reckhow, 1978d), and is presented below, that combines the uncertainty in phosphorus loading (uncertainty in $q_s$, the other variable, is assumed to be negligible) with the uncertainty in a model used to develop a loading criterion on Vollenweider's criterion axes (L vs. $q_s$). These combined uncertainties represent an estimate of total prediction uncertainty, which is expressed here as the likelihood (probability) that a lake will fall in each of the three trophic classes. Assigning a probability to a predicted classification is more consistent with our current knowledge (or ignorance), since we cannot, in fact, state with certainty what the future classification of a lake will be.

As applied in Chapter 3, the uncertainty analysis should, under certain conditions, include a term for loading error. Under other conditions, this additional loading error term would not be necessary. The following guidelines, presented first in Chapter 3, describe the conditions necessary for each error analysis approach.

1. If the model is used to assess current conditions in a lake for which the phosphorus input and hydraulic detention time are direct measurements, the level of uncertainty in these variables probably is approximately the same as the average uncertainty in these terms in the model development data set. In

that case, the uncertainty in these variables is incorporated in the model standard error, and the trophic state probabilities may be estimated using Equations A.8 and A.9 with $s_L = 0$, or using the $k = 0$ line on Figures 4.8-4.10.

2. If the model is applied to a lake for which the independent variables, particularly phosphorus loading, are indirect estimates (e.g., literature export coefficients for phosphorus loading, or direct measurements for future predictions), then it is likely that the uncertainty in these estimates is quite large. The component of the model standard error contributed by the errors in the variables is probably small in comparison. In this case, the best estimate of the total prediction uncertainty would be provided by Equations A.8-A.9, unmodified, or by Figures 4.8-4.10 (with $k$ = the fractional loading uncertainty) according to the approach described below.

The procedure, outlined and illustrated below, is entirely graphical, but may also be solved mathematically. At most, the only information required to estimate the trophic state probabilities are $L$, $q_s$, and the uncertainty in $L$. The steps are:

1. For the lake of interest, estimate $q_s$, $L$, and, when necessary, the uncertainty in $L$ (expressed as a fraction of $L$). The uncertainty in $L$ should be estimated such that about 70% of the time the actual loading will fall within $L \pm$ the uncertainty in $L$. One approach might be to estimate the maximum and minimum phosphorus loadings expected, and then assign

these loadings as $L + 2s_L$ and $L - 2s_L$, respectively (i.e., assume that the 95% confidence limits approximately represent the range). As in Chapter 3, consider Lake Charlevoix (Michigan), with data (USEPA, 1975):

$L = 0.12$ g/m$^2$-yr

$q_s = 5.24$ m/yr*

$s_L = .50\ L = kL$ = (uncertainty in L, see Chapter 3).

If desired, plot these data on Figure 4.6, with error bars representing the uncertainty in L. Figure 4.6 is a loading criterion like that proposed by Vollenweider (1975), with solid lines representing the "dangerous" and "permissible" loading levels, and dashed and dotted lines representing the uncertainty in the dangerous and permissible lines.

2. Using Figure 4.7, estimate y from $q_s$. (See the Appendix for a definition of y.) For Lake Charlevoix, y = .056. Multiply y times L. (This equals P, the predicted lake phosphorus concentration.) For Lake Charlevoix, yL = P = .0067 mg/l.

3. Using Figures 4.8, 4.9, and 4.10, the estimate of yL, and the fractional uncertainty, k, in L (if applicable): find the probability that the lake of interest should be classified as oligotrophic, mesotrophic, and eutrophic. This was done for Lake Charlevoix and:

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*Note that $q_s$ may be easily estimated, since $q_s = Q/A$, where Q = average annual volumetric water flow to lake, and A = lake surface area.

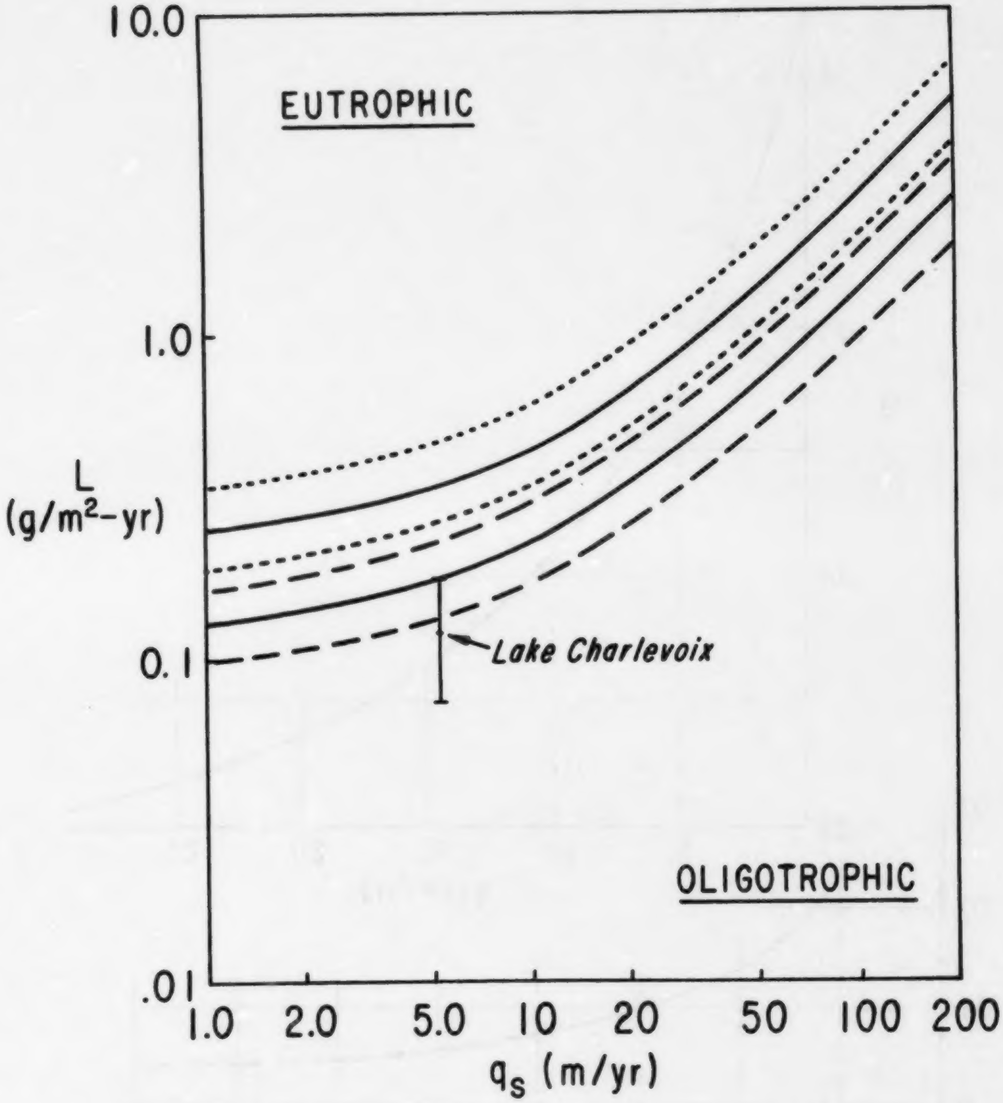

Figure 4.6 Uncertainty Estimates for the Loading Criteria on Vollenweider's 1975 Axes of L Versus $q_s$ (From Reckhow, 1978d)

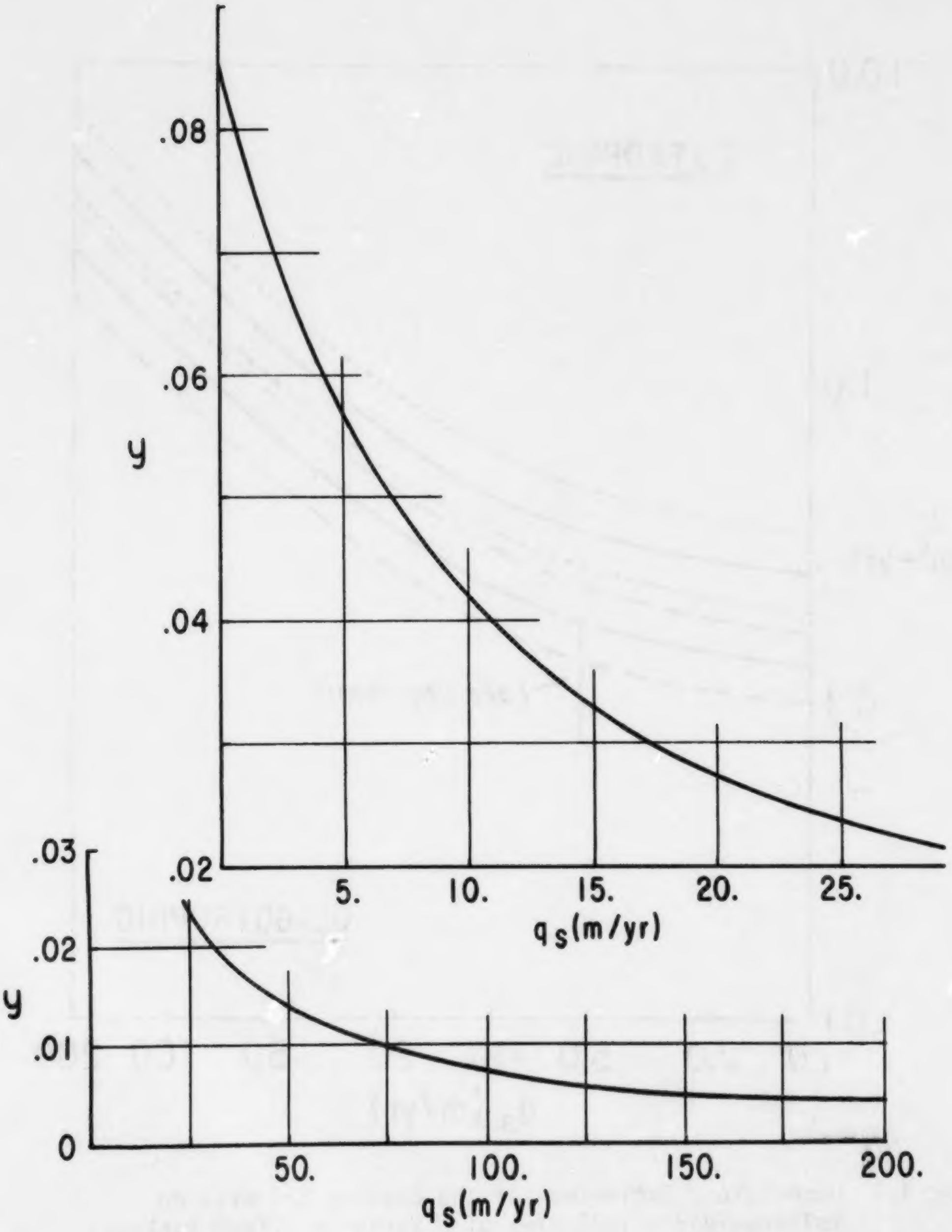

Figure 4.7 y Versus $q_s$ (From Reckhow, 1978d)

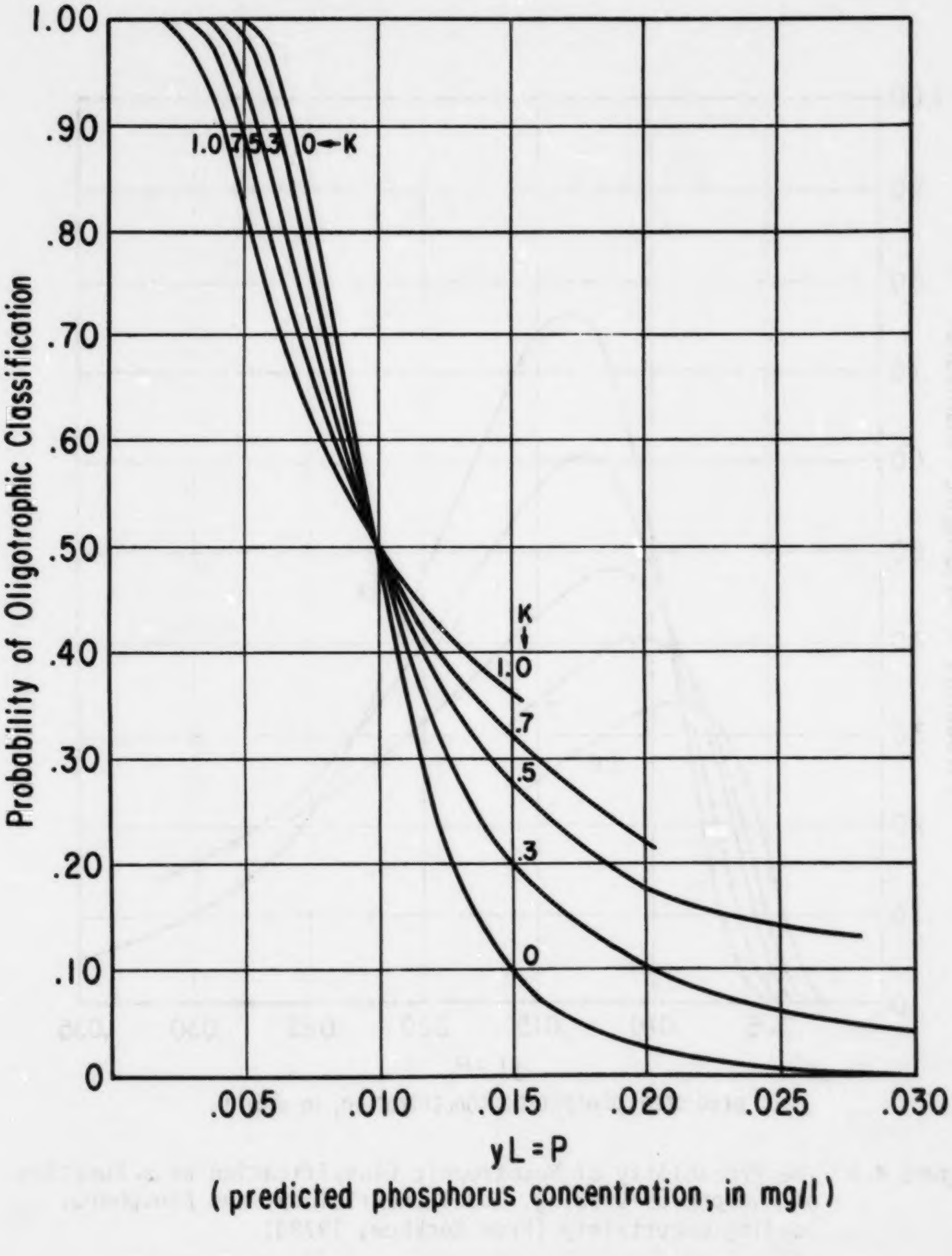

Figure 4.8 The Probability of Oligotrophic Classification as a Function of Phosphorus Loading, Model Uncertainty, and Phosphorus Loading Uncertainty (From Reckhow, 1978d)

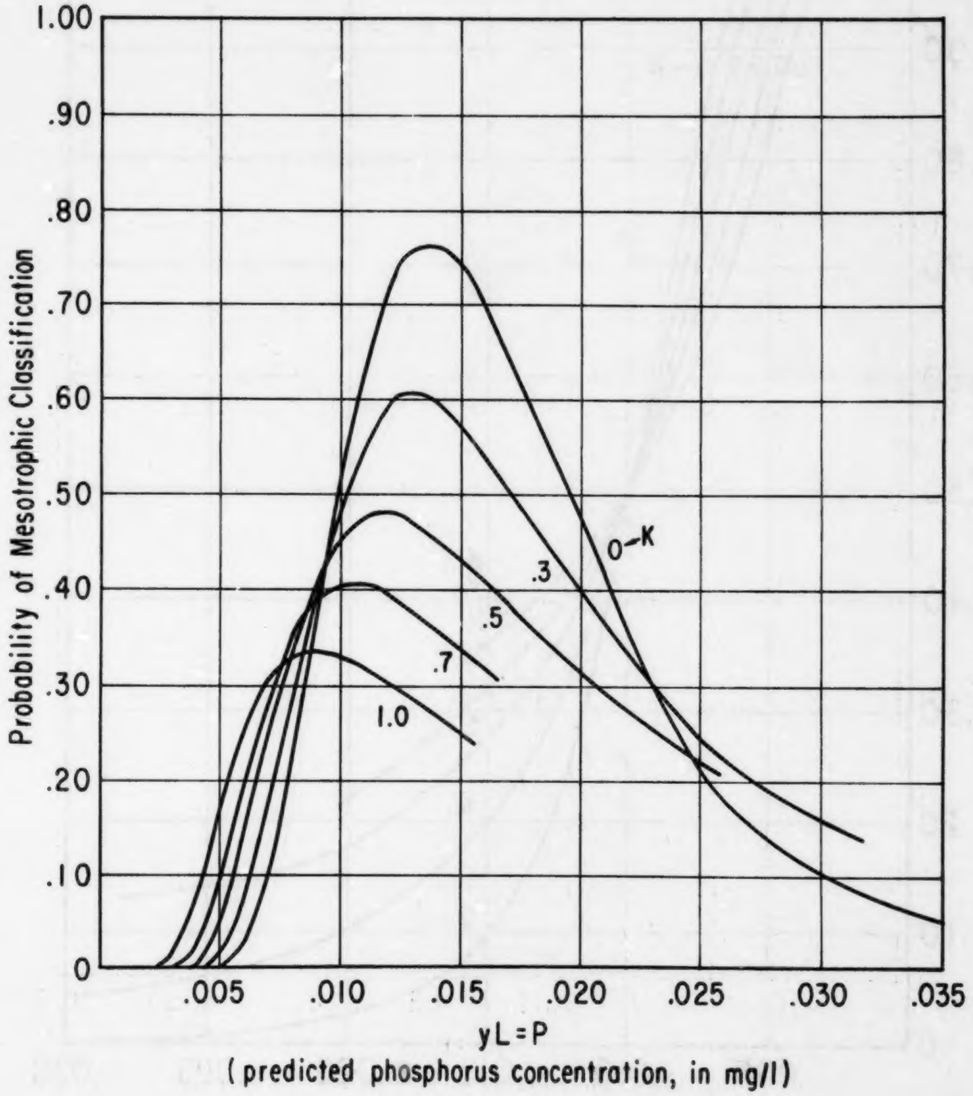

Figure 4.9 The Probability of Mesotrophic Classification as a Function of Phosphorus Loading, Model Uncertainty, and Phosphorus Loading Uncertainty (From Reckhow, 1978d)

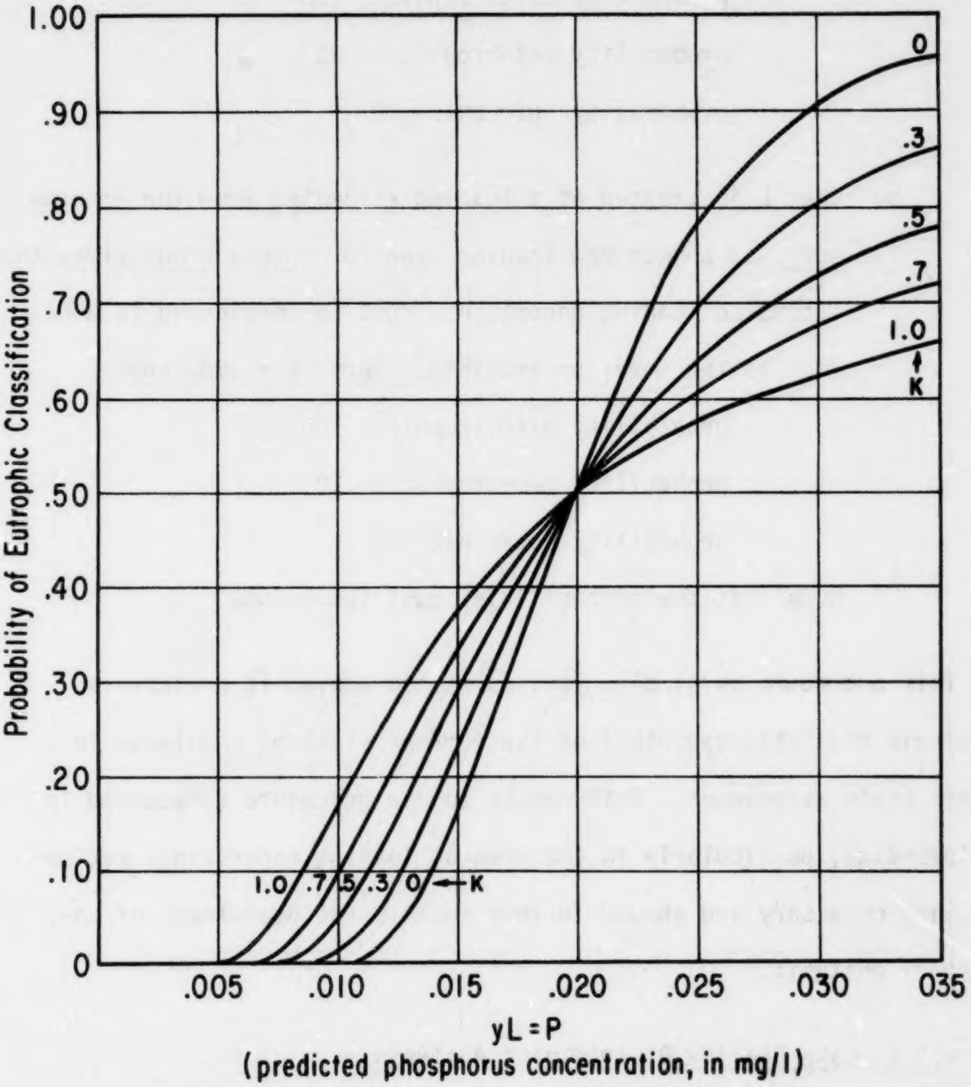

Figure 4.10 The Probability of Eutrophic Classification as a Function of Phosphorus Loading, Model Uncertainty, and Phosphorus Loading Uncertainty (From Reckhow, 1978d)

a. when L is treated as a measured loading, as part of the National Euthrophication Survey, there is no need for an additional loading error term. Thus k = 0, and:

probability oligotrophic ≃ .95
probability mesotrophic ≃ .05
probability eutrophic ≃ 0

b. when L is treated as a loading estimated from the literature, or a measured loading used for future prediction, the estimated loading uncertainty must be considered in addition to the model uncertainty. Here, k = .50, and:

probability oligotrophic ≃ .80
probability mesotrophic ≃ .20
probability eutrophic ≃ 0

Note that the probabilities must sum to one.

This procedure is simple, yet, as stated above, it presents a relatively realistic appraisal of the current state of knowledge in trophic state assessment. Refinements to the procedure (discussed in the Appendix), particularly in the area of loading uncertainty estimation, are necessary and should further improve the usefulness of uncertainty analysis.

### 4.3.5 Lake Quality Discriminant Analysis

Some of the phosphorus loading criteria discussed above rely on subjective judgement for trophic state assessment, and thus probably do not make "optimal" use of the information contained in the limnological variables (L, z, $q_s$, and $\tau$) for classifying lakes. In addition, not all of the above criteria are functionally dependent upon at least two of the geomorphology/hydrology variables $q_s$, $z$, and $\tau$, and thus, again, these criteria may not make best use of the information available.* In this section, two lake classification schemes are proposed that represent an attempt to address those expressed concerns. The technique of discriminant analysis was used to construct an optimal classification function. Discriminant analysis may be considered as analogous to regression analysis (Chapter 2), except that the dependent variable is discrete instead of continuous. For example, regression analysis may be used to develop a predictive equation for phosphorus concentration, which can assume a continuous set of values, whereas discriminant analysis may be used to develop a predictive equation for trophic state, which, as defined here, can assume only three values (oligotrophic, mesotrophic, and eutrophic). For the loading criteria described below, discriminant analysis was used to develop a trophic state classification scheme and a function, or graph, for classifying a lake as either oxic or anoxic. Some details on the development of these criteria are presented in the Appendix.

Walker (1977) used discriminant analysis on a data set of about 100 north temperate lakes in order to develop a predictive function, composed of $L$, $q_s$, and $\tau$, that best classified a lake into one of the trophic classes. Many of his lakes were from the EPA National Eutrophication Survey, and he relied upon the subjective judgment of these investigators for the initial assignment of trophic states for his data.

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*This of course, depends upon whether average lake quality is determined by two or more of these variables. Casual conclusions on that statement cannot be made at present.

This can lead to inconsistencies in the data, since different people often have different definitions of the trophic state transition points. Nonetheless, using the discriminant analysis technique, Walker developed the graph in Figure 4.11, which may be used to predict the probability, or likelihood, that a lake belongs in each of the three trophic states. This probability is a function of the variable x, where:

$$x = L\,[q_s\,(1 + .824\tau^{.454})]^{-.815} \qquad (4.7)$$

Note that x is quite similar in composition to Walker's phosphorus model, presented in Chapter 3.

Walker provides additional graphs that modify the probabilities according to the uncertainty in the independent variables. However, he notes that the technique used to derive Equation 4.7 incorporates the uncertainties in the model-development data set into the probabilities estimated using Figure 4.11. Thus, Figure 4.11 may be used without additional uncertainty estimates for lakes for which the data collection and assessment techniques are similar to those in Walker's model-development data set (primarily National Eutrophication Survey lakes).* This is true for the Lake Charlevoix data, for example. Thus, application of Walker's loading criterion yields the following predictions for Lake Charlevoix ($\tau$ = 3.2 yrs):**

---

*Note that this is the same criterion used for the error analysis discussed in Chapter 3 and Section 4.3.4.

**The major reason for the difference between the probabilities estimated by Walker's criterion and those estimated by Reckhow's uncertainty analysis approach in section 4.3.4 is expressed in the difference in phosphorus concentration predictions (.0067 mg/l for Reckhow's approach and .0096 mg/l for Walker's model in Chapter 3). Since the oligotrophic/mesotrophic transition point is often expressed as a phosphorus concentration of .010 mg/l, it is understandable that a slight difference in predicted phosphorus concentration at those levels can lead to a large difference in trophic state probabilities.

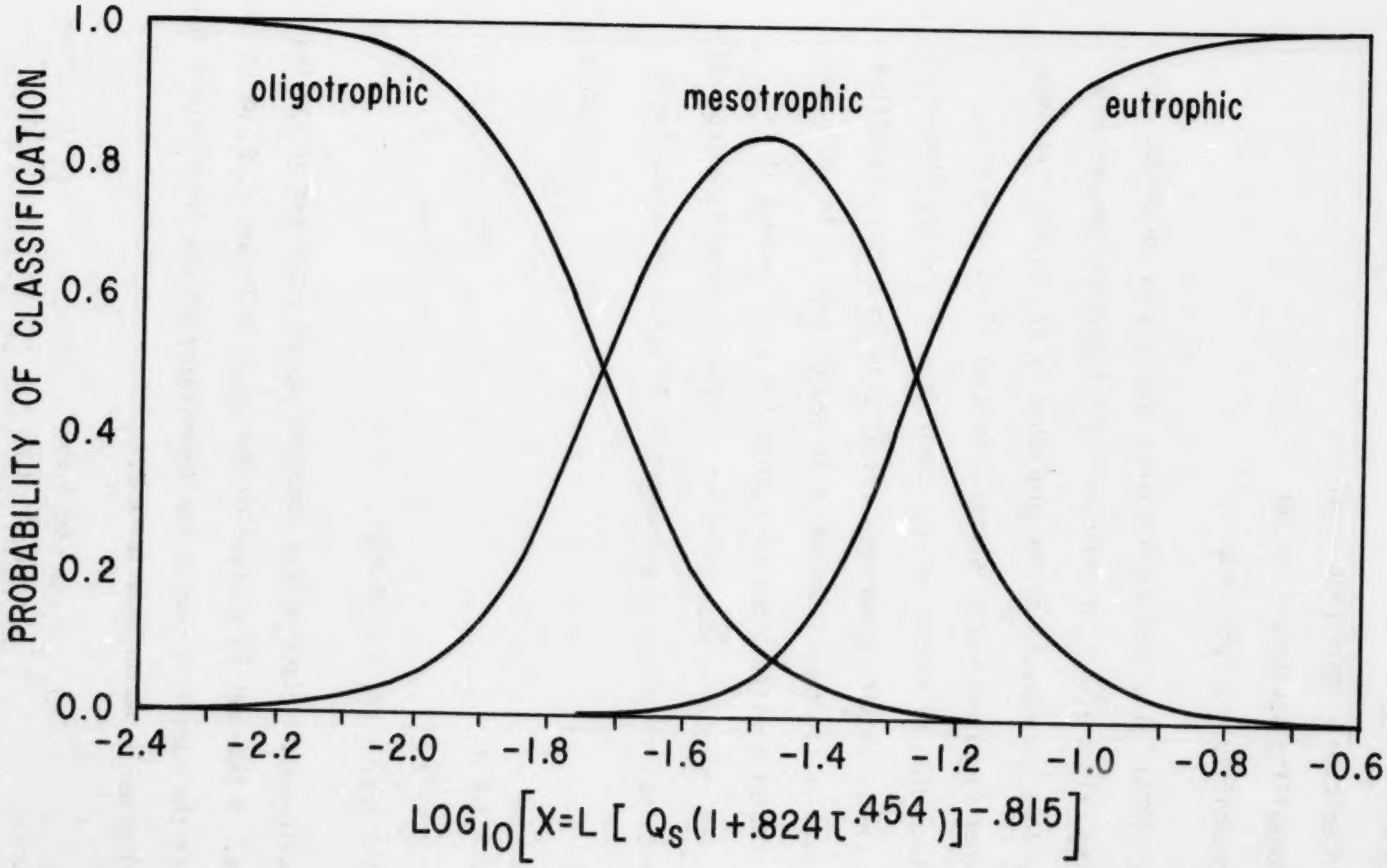

Figure 4.11 Walker's Loading Criterion Assessing the Probabilities of Trophic Classification (From Walker, 1977)

$\log x = -1.82$

probability oligotrophic ≈ .67

probability mesotrophic ≈ .33

probability eutrophic ≈ 0

Reckhow (1978a) also used discriminant analysis to construct phosphorus loading criteria, but in addition, he attempted to remove some subjectivity from the classification procedure by basing his technique on well-defined dissolved oxygen states. Specifically, lakes were classified according to oxic or anoxic conditions in the hypolimnion, such that a single anoxic measurement resulted in an anoxic classification. The oxic-anoxic transition was selected because it is well-recognized as a water quality transition point (in part because it is associated with phosphorus adsorption or release from the lake sediments). The resultant loading criterion was based on 55 north temperate lakes with:

$z > 3.0$ m

$\tau > 0.25$ yr

$1.0 \text{ m/yr} < z/\tau < 50.0 \text{ m/yr}$

This oxic-anoxic discriminant function may be expressed or applied in two ways. A lake may be plotted on the graph in Figure 4.12, which is a plot of the numerator versus the denominator of the discriminant function (d.f.) given below in Equation 4.8.

$$\text{d.f.} = \frac{10^{4.68} L^{1.95}}{z^{2.58} q_s^{1.85}} \qquad (4.8)$$

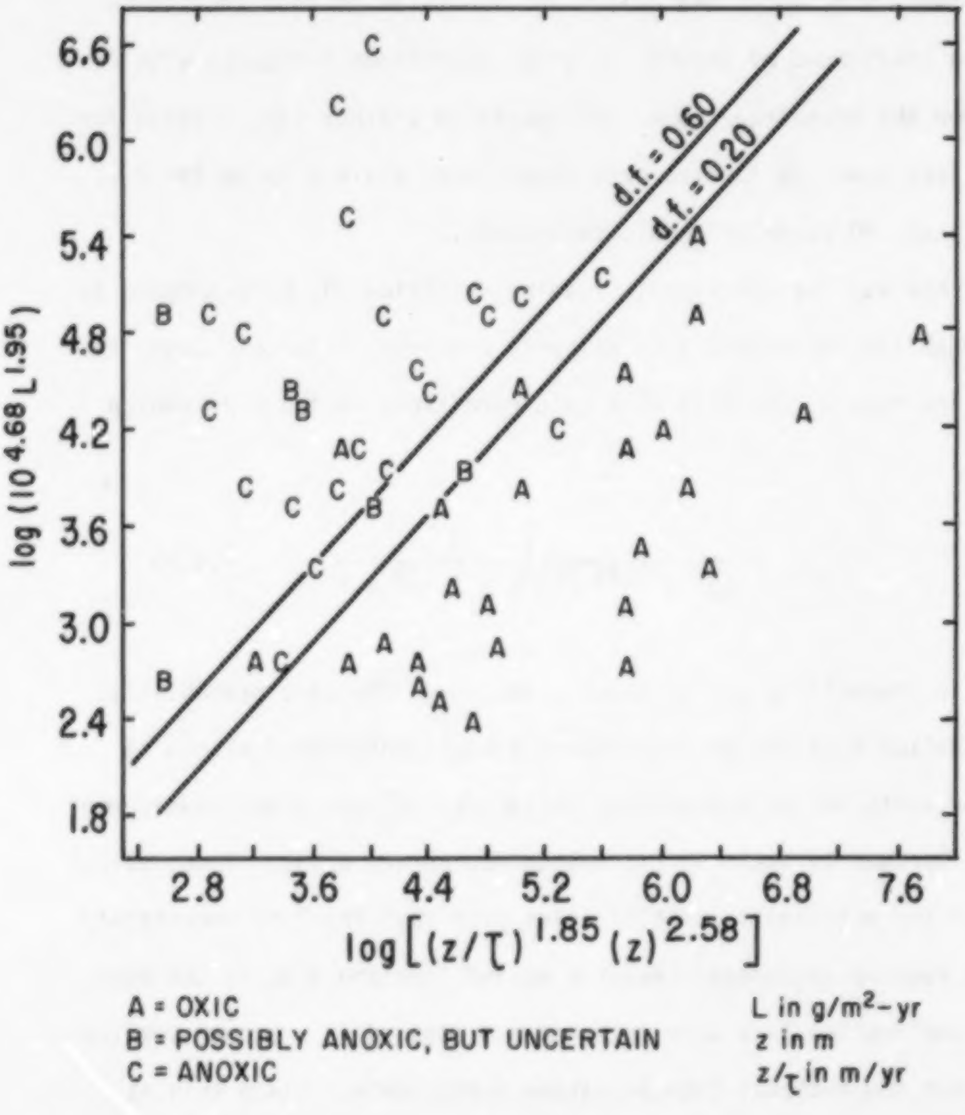

Figure 4.12 The Oxic-Anoxic Discriminant Function (From Reckhow, 1978a)

The lakes that were used to develop this discriminant function are plotted in Figure 4.12. Six of those lakes could not confidently be classified as either oxic or anoxic, and thus they help to define a "transition zone" between d.f. = .20 and d.f. = .60. Outside of this transition zone, the likelihood of anoxic, or oxic, conditions increases with distance from the transition zone. Expressed in another way, a value for d.f. of less than .20 implies oxic conditions, while a value for d.f. of more than .60 suggests anoxic conditions.

Another way the oxic-anoxic loading criterion may be expressed is in a probability statement like Walker's criterion. In this case, the probability that a lake will have oxic conditions in the hypolimnion is:

$$P_{oxic} = \frac{1}{10^5 \, z^{-2.49} \, L^{2.00} \, q_s^{-1.78} + 1} \qquad (4.9)$$

The anoxic probability is, of course, one minus the oxic probability. As with Walker's criterion, Equation 4.9 has incorporated within it the uncertainty in the independent variables from the model-development data set (primarily National Eutrophication Survey data). Thus, calculation of the oxic-anoxic probabilities with that level of uncertainty does not require additional analysis beyond Equation 4.9, if the data for the application lake were gathered and reduced in a manner similar to that for the National Eutrophication Survey data. Since this is the case for Lake Charlevoix ($z$ = 16.8 m), the probability of oxic conditions, in consideration of uncertainty in $L$, $q_s$, and $z$, is:

$$P_{oxic} = \frac{1}{10^5(16.8)^{-2.49}(.12)^{2.00}(5.24)^{-1.78} + 1}$$

$$P_{oxic} = .94$$

$$P_{anoxic} = .06$$

The oxic probabilities were assessed, using Equation 4.9, for Lakes Washington and Sammamish before and after nutrient diversion projects. After these diversions, noticeable improvements in water quality were observed in Lake Washington but not in Lake Sammamish. Figure 4.13 is a plot of the oxic probabilities, and it represents evidence supporting one reason why different results developed from the diversion projects. While the post-diversion phosphorus loading resulted in a high likelihood of oxic conditions in Lake Washington, the likelihood of oxic conditions associated with the post-diversion phosphorus loading in Lake Sammamish was only about 0.5. This implies that phosphorus loading must be reduced more, to substantially decrease the likelihood of anoxic conditions and improve water quality. Thus, Equation 4.9 may be used to assess the impact of changes in phosphorus loading on water quality.

## 4.4 Summary

A number of trophic state classification schemes have been evaluated, and it was found that most approaches convey essentially the same information. The selection of the best or most appropriate index, for a given application, probably should be a function of familiarity, perceived credibility, ease of understanding, and robust-

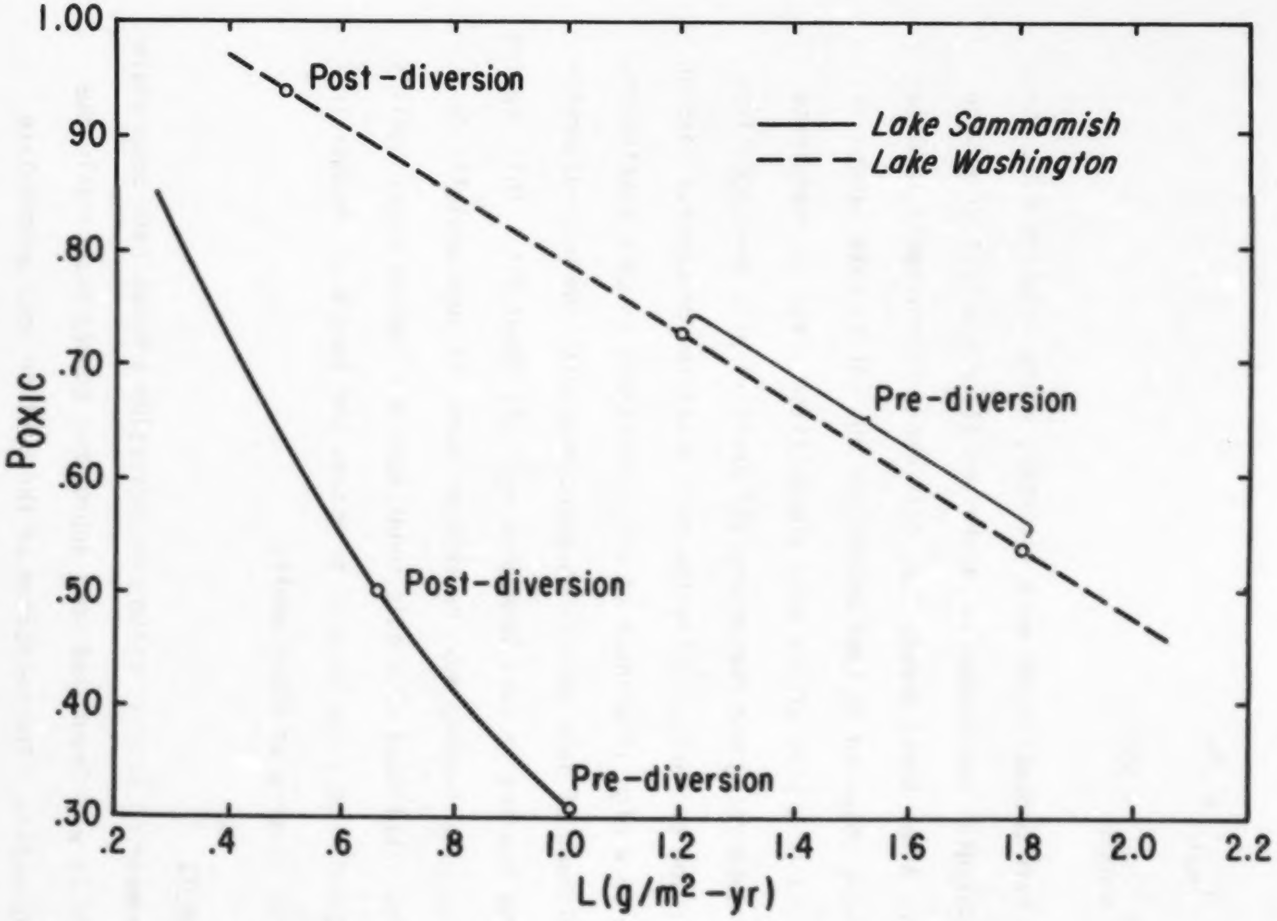

Figure 4.13 Lake Sammamish and Lake Washington: Probability of Oxic Conditions ($P_{oxic}$) vs. Annual Areal Phosphorus Loading (L). (From Reckhow, 1978a)

ness under unusual limnological conditions. The index proposed by Carlson is suggested here as the best choice on the basis of the latter two points.

Trophic state indicators have limited value in lake watershed management and planning, however, since they evaluate only current conditions. Nutrient, particularly phosphorus, loading criteria, are required for that purpose. Therefore, several of the early loading criteria were evaluated, and while these criteria are useful and often successful, they can be improved. Improvements that were suggested, and that have led to the development of new loading criteria, deal with optimality, objectivity, and uncertainty. In particular, recent proposals by Reckhow and Walker have focused on consideration of these concepts in the construction of new phosphorus loading criteria. These new criteria should generally convey more reliable information on the effect of phosphorus loading than earlier criteria. Refinements, particularly in the area of uncertainty analysis, should further improve the usefulness of these approaches.

As a final note, it should be stated that the loading criteria discussed here are no more or no less accurate than the phosphorus models discussed in Chapter 3. In fact, as pointed out in this chapter, many of the loading criteria are simply graphical representations of a phosphorus model. They are expressed in graphical form for convenience and for display purposes, but they convey no more information than the phosphorus models. Thus the same warning given in Chaper 3 is applicable here; apply the loading criteria with caution and a good understanding of the documented uses, and limitations, of each technique.

# APPENDIX

## DERIVATION AND EXPLANATION OF THE PROCEDURE USED TO APPLY UNCERTAINTY ANALYSIS TO VOLLENWEIDER'S LOADING CRITERION

In order to estimate the uncertainty associated with the application of phosphorus loading criteria on Vollenweider's criterion axes (L vs. $q_s$), a model is needed that provides the equation of the trophic state transition lines. It was assumed here that trophic states are determined by the total phosphorus concentration in a lake. Further, it was assumed that the transition points are those proposed by the EPA (1974) in Table 4.2. These are:

| Trophic State | Total Phosphorus Concentration (mg/l) |
|---|---|
| oligotrophic | < .010 |
| mesotrophic | .010 - .020 |
| eutrophic | >.020 |

A model was developed, for north temperate lakes, to predict the phosphorus concentration at the transition points. The uncertainty associated with this model, which is one component of the total prediction uncertainty, was then used to construct error bars (expressed as a function of uncertainty in L) on the loading criterion, Figure 4.6. This is intended to graphically illustrate the degree of uncertainty in the model that specifies where, on this graph of L versus $q_s$, the trophic state transition lines, associated with phosphorus concentrations of .010 mg/l and .020 mg/l, must lie.

The model-development data set lakes are characterized by:

| Variable | Minimum | Maximum |
|---|---|---|
| $P$ | .004 mg/l | .135 mg/l |
| $L$ | .07 $g/m^2$-yr | 31.4 $g/m^2$-yr |
| $q_s$ | 1.23 m/yr | 187. m/yr |

The lakes are assumed to be completely mixed, and the phosphorus concentration represents an "average" value (no distinction among spring, growing-season, or annual averages).

The model used to predict the position of the trophic state transition lines is of the form:

$$P = \frac{L}{v_s + q_s} \tag{A.1}$$

where

$v_s$ = apparent settling velocity (m/yr).

Using weighted least squares regression, $v_s$ was found to be a linear function of $q_s$:

$$v_s = 11.6 + 0.2\, q_s \tag{A.2}$$

Therefore, the phosphorus model becomes:

$$P = \frac{L}{11.6 + 1.2\, q_s} = yL \tag{A.3}$$

where

$$y = 1/(11.6 + 1.2\ q_s) \qquad \text{(A.4)}$$

The model standard error, $s_m$, was estimated from:

$$s_m = \sqrt{\frac{\sum_{i=1}^{n} (P_{ob} - P_e)^2}{n - 2}} \qquad \text{(A.5)}$$

where

$P_{ob}$ = observed phosphorus concentration (mg/l)

$P_e$ = estimated (predicted) phosphorus concentration (mg/l)

$n$ = 47 lakes

For the log-transformed model, $s_m = .128$. In addition, the parameter error contribution to the total uncertainty was found to be negligible in comparison to $s_m$, and thus it was dropped from further consideration.

Therefore, the error lines in Figure 4.6 represent the standard error of the model ($s_m$), and were estimated in the following way. The equation for the trophic state transition lines is:*

$$L = (11.6 + 1.2\ q_s)\ (P) \qquad \text{(A.6)}$$

where

$P$ = .010 mg/l, .020 mg/l.

The equation for the error lines then becomes* (after modifications, necessary because the model was estimated in the log-transformed state):

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*An approximation, since the regression of x on y does not equal the regression of y on x.

$$L = (11.6 + 1.2\ q_s)\ (10^{\log P \pm s_m}) \qquad \text{(A.7)}$$

where

$P$ = .010 mg/l, .020 mg/l

$s_m$ = .128

Figures 4.8, 4.9, and 4.10 were constructed by summing the squared values of the uncertainty estimates for:

1. the model (the standard error of the estimate), $s_m$.
2. the total uncertainty component from the fractional uncertainty in phosphorus loading, $s_L$.

The independent variable ($L, q_s$) uncertainty was assumed to be a function of loading uncertainty alone. Therefore, the total prediction uncertainty ($s_T$) was estimated as:

$$s_T^2 = s_m^2 + s_L^2 \qquad \text{(A.8)}$$

The curves in Figures 4.8 - 4.10 were calculated, assuming normally-distributed errors, using the standard normal deviate, $z_n$:

$$z_n = \frac{P_{t.s.} - P_e}{s_T} \qquad \text{(A.9)}$$

where

$P_{t.s.}$ = .010 mg/l, .020 mg/l; the phosphorus concentrations associated with the trophic state transitions.

$P_e$ = estimated phosphorus concentration, using Equation A.3.

Note that the probabilities were estimated by assuming that the model error is associated with $P_e$, not $P_{t.s.}$ (i.e., the transition lines are "true" values, with no uncertainty, while the estimated lake concentration has model and loading uncertainty associated with it).

In summary, two comments must be made on the limitations on the use of this procedure:

1. Application lakes should have characteristics not unlike those of the model development data set discussed earlier.

2. This must be considered an introduction to the use of uncertainty analysis in lake quality assessment. Since it is a "first-cut," there are no established procedures for estimating loading uncertainty, and there is even confusion as to the meaning of loading uncertainty. As expressed here, the loading uncertainty is determined such that almost 70% of the time that the loading uncertainty is estimated, the "true" loading (under steady-state conditions) will lie within the estimated loading ± the loading uncertainty. Clearly, this is difficult to estimate, given natural fluctuations, sampling limitations, human error, and human inexperience.* Thus, more work and experience are needed before a successful uncertainty analysis approach is developed that has the same meaning for both the model developer and the model user.

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*An additional problem develops when the phosphorus loading uncertainty nears ± 100% and is assumed to be normally distributed. When this occurs, there is a nonzero probability that the lake phosphorus concentration will be less than zero. For this reason, some of the lines in Figures 4.8 - 4.10 are shorter (horizontally) than others, and for this reason more work is needed on the distributional aspects of the loading uncertainty.

## LAKE QUALITY DISCRIMINANT FUNCTIONS

Discriminant analysis is a useful procedure for constructing a predictive relationship for a dependent variable expressed in a small number of discrete states. A heuristic analogy was drawn in Section 4.3.5 between discriminant analysis and regression analysis. That is, regression analysis is most often used to develop predictive equations for a continuous dependent variable (like phosphorus concentration), whereas discriminant analysis may be used to develop predictive relations for a discretized dependent variable (like trophic state).

For the oxic-anoxic discriminant function, the following discriminant function was presented:

$$\text{d.f.} = \frac{10^{4.68}\, L^{1.95}}{z^{2.58}\, q_s^{1.85}} \qquad \text{(A.10)}$$

This function is actually the first principal component, and it maximizes the explained variance or discrimination between classes. Since it is a single component, it maximizes the group separation on a single axis.

In order to be consistent with other graphical classification criteria, a plot of this function is made, in Figure 4.12, with the log of the numerator on the y-axis and the log of the denominator on the x-axis. A transition zone is defined, with the help of the "uncertain" lakes, to convey the notion that the transition from oxic to anoxic conditions (or vice versa) cannot be specified with certainty.

The same information found in the discriminant function may be re-expressed in terms of classification functions which lead to classification probabilities. The classification function, c.f., is converted to a classification probability estimate using:

$$P_{oxic} = \frac{1}{e^{(c.f.)} + 1} \qquad (A.11)$$

with

$$P_{anoxic} = 1 - P_{oxic} \qquad (A.12)$$

When the classification function is substituted into Equation A.11, the oxic probability becomes:

$$P_{oxic} = \frac{1}{10^{5}\ z^{-2.49}\ L^{2.00}\ q_s^{\ -1.78} + 1} \qquad (A.13)$$

# CHAPTER 5

# FINAL COMMENTS

At this point, it seems appropriate to offer some final observations on the application and development of quantitative techniques like those discussed in this manual. First, some thoughts on applications are presented below:

1. The estimation of data uncertainty and distribution variance probably creates uneasiness in the minds of many users. Nonetheless, the importance of these estimates should be clear from the discussion in this manual. Further, as more investigators think about these terms, develop appropriate estimates, and present them in the literature, this task should become easier.

2. The models need to be considered within the broad context of lake management. That is, the models merely provide information on the impact of watershed characteristics and activities on lake quality. This information is just one input to the analysis of lake management strategies. We need to know, in addition, the costs and benefits associated with suggested management strategies for the lake of study. Further, with explicit consideration of uncertainty, we should try to assess the risk of overestimation and underestimation of water quality

goals. It is important, therefore, not to rely too heavily on the lake models and projected lake trophic states for the determination of policy, since policy is not simply a technical issue.

Finally, the list below presents proposed research to improve the usefulness of simple lake models coupled with uncertainty analysis:

1. A better procedure is needed for the treatment of phosphorus loading error. One approach would involve the removal of the loading error contribution, attributed to the "average" loading error in the model development data set, from the model standard error.* Then, all applications of this approach would require this revised model standard error and an estimate of the loading error for the application lake.

2. Another way to eliminate the problem of phosphorus loading error would be to develop a model that bypasses phosphorus loading. That is, phosphorus loading is actually an "intervening variable" between the decision variable, phosphorus concentration, and the control variables, watershed characteristics and activities. When enough data are available on watershed land use for several lakes, modeling work should probably be undertaken to directly relate the watershed to the lake quality. If these models can be successfully developed, they should be easier to apply since data may be avail-

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*This would also necessitate re-estimation of the model coefficients to correct any bias attributed to the variable error.

able from land use maps, and since independent variable error could be relatively small.

3. This author is currently involved in research on the following topics in lake modeling:
    a. A procedure is being developed (Chapra and Reckhow, 1978) that expresses Vollenweider's 1976 trophic state criterion in probabilistic terms.

    b. Work is underway (Reckhow, 1978e) to develop a model that relates nutrient loading, lake hydrology, and lake geomorphology to the probability of a lake water quality standard violation. Many lake water quality standards are not expressed in terms of the traditional model dependent variables, like nutrient concentrations. Therefore, this research defines a more common water quality standard variable, minimum dissolved oxygen level, as the dependent variable. Thus, the proposed model will predict the likelihood of a standard violation for dissolved oxygen, as a function of the independent variables listed above.

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# STATISTICAL SYMBOLS

| Symbol | Definition |
|---|---|
| $\overline{x}$ | sample mean |
| $s$ | sample standard deviation |
| $Y_{pred.}$ | predicted Y value |
| $Y_{obs.}$ | observed Y value |
| d.f. | degrees of freedom |
| $t$ | student's t |
| $n$ | number of samples |
| $x_o$ | a particular value of x |
| $\overline{x}_i$ | the mean for group i |
| $n_i$ | the number of samples in group i |
| $s^2$ | sample of variance |
| $d$ | precision, or uncertainty, of sampling design |
| $S_{\overline{x}}$ | standard error of the mean |
| $r$ | correlation coefficient |
| $R^2$ | coefficient of determination |
| $s_m$ | standard error of the estimate |
| $C(n)$ | cost of sampling |
| $C_o$ | fixed cost of sampling |
| $C_i$ | cost of sampling stratum i |
| c.v. | coefficient of variation |
| $W_i$ | sampling design weighting factor for stratum i |

## SYMBOLS FOR LAKE MODELS

| Symbol | Definition | Units |
|---|---|---|
| $P$ | lake phosphorus concentration | mg/l |
| $V$ | lake volume | $10^6 m^3$ |
| $M$ | annual mass rate of phosphorus inflow to lake | $10^3 kg/yr$ |
| $L$ | annual areal phosphorus loading | $g/m^2$-yr |
| $z$ | lake mean depth | m |
| $\tau$ | hydraulic detention time | yr |
| $\sigma$ | sedimentation coefficient | $yr^{-1}$ |
| $Q$ | annual volume rate of water inflow to lake | $10^6 m^3/yr$ |
| $P^o$ | average lake phosphorus concentration to time t = 0 | mg/l |
| $A$ | lake surface (bottom) area | $km^2$ |
| $v_s$ | apparent settling velocity | m/yr |
| $q_s = z/\tau = Q/A$ | surface overflow rate | m/yr |
| $R_p$ | fraction of influent phosphorus retained in a lake | |
| $P_o$ | average outflow phosphorus concentration | mg/l |
| $P_{in} = \frac{L\tau}{z}$ | average influent phosphorus concentration | mg/l |
| $M_I$ | annual mass influx of phosphorus | |
| $M_o$ | annual mass efflux of phosphorus | |
| $\phi$ | annual net flux of phosphorus to the sediments | |
| $R_{KD}$ | Kirchner and Dillon empirical estimate for $R_p$ | |
| $R_{LM}$ | Larsen and Mercier empirical estimate for $R_p$ | |

| Symbol | Definition | Units |
|---|---|---|
| $s'_T$ | total prediction error | mg/l |
| $s'_{iv}$ | input variable error (untransformed) contribution to total error | mg/l |
| $s_T$ | total error (logarithmic form) | |
| $s_p$ | parameter error contribution to the total error (log form) | |
| $s_m$ | standard error of the estimate for the model (log form) | |
| $s_{iv}$ | input variable error (log form) contribution to total error | |
| $P_e$ | predicted phosphorus concentration | mg/l |
| $P_{true}$ | the "true" total phosphorus concentration | mg/l |
| k | a constant | |
| $L_c$ | critical phosphorus loading | $g/m^2$-yr |
| $P_c$ | critical phosphorus concentration | mg/l |
| $P_{ob}$ | observed phosphorus concentration | mg/l |
| log | base 10 logarithm | |
| ln | base e logarithm | |